B5.3 Viscous Flow: Sheet 1

Q1 Vector identities and the divergence theorem.

(a) For a differentiable scalar field $f(\mathbf{x})$ and a differentiable vector field $\mathbf{G}(\mathbf{x}) = G_i(\mathbf{x})\mathbf{e}_i$, the differential operators introduced in "Calculus in Three-Dimensions and Applications" may be written using the summation convention in the form

$$\boldsymbol{\nabla} f = \mathbf{e}_i \frac{\partial f}{\partial x_i}, \quad \boldsymbol{\nabla} \cdot \mathbf{G} = \frac{\partial G_j}{\partial x_j}, \quad \boldsymbol{\nabla} \wedge \mathbf{G} = \mathbf{e}_k \wedge \frac{\partial \mathbf{G}}{\partial x_k}, \quad (\mathbf{u} \cdot \boldsymbol{\nabla})\, f = u_l \frac{\partial f}{\partial x_l}, \quad \boldsymbol{\nabla}^2 f = \frac{\partial^2 f}{\partial x_m \partial x_m},$$

where $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are unit vectors along the axes Ox_1, Ox_2, Ox_3. Using these definitions, the vector identity

$$(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$$

and the summation convention, prove the following identities for any differentiable scalar field f and any differentiable vector fields $\mathbf{u}$ and $\mathbf{v}$:

(i) $\boldsymbol{\nabla} \cdot (f\mathbf{u}) = (\mathbf{u} \cdot \boldsymbol{\nabla})f + f(\boldsymbol{\nabla} \cdot \mathbf{u})$;

(ii) $(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = \boldsymbol{\nabla}\left(\frac{1}{2}|\mathbf{u}|^2\right) + (\boldsymbol{\nabla} \wedge \mathbf{u}) \wedge \mathbf{u}$;

(iii) $\boldsymbol{\nabla} \wedge (\mathbf{u} \wedge \mathbf{v}) = (\boldsymbol{\nabla} \cdot \mathbf{v})\mathbf{u} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{u} - (\boldsymbol{\nabla} \cdot \mathbf{u})\mathbf{v} - (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{v}$;

(iv) $\boldsymbol{\nabla}^2\mathbf{u} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) - \boldsymbol{\nabla} \wedge (\boldsymbol{\nabla} \wedge \mathbf{u})$.

(b) The faces of a tetrahedron lie in the planes $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ and $\mathbf{a} \cdot \mathbf{x} = 1$, where $\mathbf{a} = a_i\mathbf{e}_i$ is a unit vector such that $a_j > 0$ for $j = 1, 2, 3$. Let A_j be the area of the face in the plane $x_j = 0$ and let A be the area of the slanted face with unit normal $\mathbf{a}$. State the divergence theorem and by applying it to $\mathbf{e}_j$, show that $A_j = a_jA$.

Q2 Euler's identity and Reynolds' transport theorem. Suppose that $\mathbf{x} = (x_1, x_2, x_3)$ denotes the Eularian coordinates of a fluid particle with Lagrangian coordinates $\mathbf{X} = (X_1, X_2, X_3)$, *i.e.*

$$\frac{\mathrm{D}\mathbf{x}}{\mathrm{D}t} = \mathbf{u}, \quad \text{with } \mathbf{x} = \mathbf{X} \text{ at } t = 0,$$

where $\frac{\mathrm{D}}{\mathrm{D}t} = \left.\frac{\partial}{\partial t}\right|_{\mathbf{X}}$ is the convective derivative and $\mathbf{u} = (u_1, u_2, u_3)$ is the velocity. Let J be the Jacobian

$$J = \frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)} = \varepsilon_{ijk} \frac{\partial x_1}{\partial X_i} \frac{\partial x_2}{\partial X_j} \frac{\partial x_3}{\partial X_k},$$

where ε_{ijk} is the Levi-Civita symbol.

(a) (i) Use the chain rule for differentiation to show that

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial x_n}{\partial X_i}\right) = \frac{\partial u_n}{\partial x_m} \frac{\partial x_m}{\partial X_i}.$$

(ii) Hence show that

$$\frac{\mathrm{D}J}{\mathrm{D}t} = \frac{\partial u_1}{\partial x_m} \frac{\partial(x_m, x_2, x_3)}{\partial(X_1, X_2, X_3)} + \frac{\partial u_2}{\partial x_m} \frac{\partial(x_1, x_m, x_3)}{\partial(X_1, X_2, X_3)} + \frac{\partial u_3}{\partial x_m} \frac{\partial(x_1, x_2, x_m)}{\partial(X_1, X_2, X_3)}.$$

(iii) Using the fact that a determinant is zero if it has repeated rows, deduce Euler's identity

$$\frac{\mathrm{D}J}{\mathrm{D}t} = J\, \boldsymbol{\nabla} \cdot \mathbf{u}.$$

(b) Let $f(\mathbf{x}, t)$ be a differentiable function of position $\mathbf{x}$ and time t. Show that the rate of change of f following a material fluid element is given by

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\partial f}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})f.$$

(c) By writing

$$\iiint_{V(t)} f(\mathbf{x}, t)\, \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 = \iiint_{V(0)} f(\mathbf{x}, t)J\, \mathrm{d}X_1\mathrm{d}X_2\mathrm{d}X_3,$$

prove Reynolds Transport Theorem that

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_{V(t)} f\, \mathrm{d}V = \iiint_{V(t)} \frac{\partial f}{\partial t} + \boldsymbol{\nabla} \cdot (f\mathbf{u})\, \mathrm{d}V$$

for any continuously differentiable function $f(\mathbf{x}, t)$, where $V(t)$ is a material volume of fluid and $\mathrm{d}V = \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3$.

(d) By applying the principle of conservation of mass to a material volume $V(t)$, derive the continuity equation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0$$

for a compressible fluid with density ρ, stating any assumptions that you make about the smoothness of ρ and $\mathbf{u}$. Hence show that a corollary of Reynolds' transport theorem for a compressible fluid is that

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_{V(t)} \rho F \, \mathrm{d}V = \iiint_{V(t)} \rho \frac{\mathrm{D}F}{\mathrm{D}t} \, \mathrm{d}V$$

for any continuously differentiable function $F(\mathbf{x}, t)$.

Q3 The stress vector and stress tensor.

(a) Define carefully the stress vector $\mathbf{t}(\mathbf{x}, t, \mathbf{n})$ and the stress tensor $\sigma_{ij}(\mathbf{x}, t)$. In the remainder of this question you may assume that

$$\mathbf{t}(\mathbf{x}, t, -\mathbf{n}) = -\mathbf{t}(\mathbf{x}, t, \mathbf{n}).$$

What is the physical significance of this expression?

(b) Fluid flows outside a rigid solid rectangular box $R = \{\mathbf{x} : 0 < x_j < j \text{ for } j = 1,\, 2,\, 3\}$. Let $\mathbf{F}_i^{\pm}(t)$ denote the surface force exerted by the fluid on the face with outward pointing unit normal $\pm\mathbf{e}_i$. Explain briefly why

$$\mathbf{F}_3^{+}(t) = \int_0^2 \int_0^1 \mathbf{t}(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + 3\mathbf{e}_3, t, \mathbf{e}_3) \, \mathrm{d}x_1 \mathrm{d}x_2.$$

Write down similar expressions for the surface force exerted by the fluid on each of the other faces of the box. In terms of the stress tensor σ_{ij}, write down an expression for the x_1-component of the net surface force exerted by the fluid on the box.

Q4 Derivation of the incompressible Navier-Stokes equations.

(a) Newton's second law for a material volume $V(t)$ that has boundary $\partial V(t)$ with outward unit normal $\mathbf{n}$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_{V(t)} \rho \mathbf{u} \, \mathrm{d}V = \iint_{\partial V(t)} \mathbf{t}(\mathbf{n}) \, \mathrm{d}S + \iiint_{V(t)} \rho \mathbf{F} \, \mathrm{d}V,$$

where ρ is the density, $\mathbf{u} = u_i \mathbf{e}_i$ is the velocity and $\mathbf{F} = F_i \mathbf{e}_i$ is an external body force acting per unit mass. Explain the physical significance of each term in this expression.

(b) Use the corollary to Reynolds' transport theorem, Cauchy's stress theorem and the divergence theorem to derive Cauchy's momentum equation in the form

$$\rho \frac{\mathrm{D}u_i}{\mathrm{D}t} = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho F_i.$$

(c) Use the continuity equation to show that a flow is incompressible if and only if

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0.$$

(d) Define the rate-of-strain tensor e_{ij}. State sufficient conditions for an incompressible fluid to be Newtonian, that is

$$\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij},$$

where p is the pressure and μ is the viscosity.

(e) For an incompressible, constant viscosity, Newtonian fluid, deduce the Navier-Stokes equations in the form

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} \right) = -\boldsymbol{\nabla} p + \mu \boldsymbol{\nabla}^2 \mathbf{u} + \rho \mathbf{F}, \quad \boldsymbol{\nabla} \cdot \mathbf{u} = 0. \tag{1}$$

Q5 The vorticity-transport equation.

(a) Show that the momentum equation (1) may be written in the form

$$\frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\omega} \wedge \mathbf{u} + \boldsymbol{\nabla}\left(\frac{p}{\rho} + \frac{1}{2}|\mathbf{u}|^2\right) = -\nu \boldsymbol{\nabla} \wedge \boldsymbol{\omega} + \mathbf{F},$$

where $\boldsymbol{\omega} = \boldsymbol{\nabla} \wedge \mathbf{u}$ is the vorticity and $\nu = \mu/\rho$ is the coefficient of kinematic viscosity.

(b) Suppose that $\mathbf{F}$ is a conservative vector field. By taking the curl of the momentum equation as written above derive the vorticity-transport equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\,\mathbf{u} = \nu \boldsymbol{\nabla}^2 \boldsymbol{\omega}.$$

(c) Suppose that in addition the flow is two-dimensional with velocity $\mathbf{u} = u(x,y,t)\mathbf{i} + v(x,y,t)\mathbf{j}$.

(i) Show that the vorticity $\boldsymbol{\omega} = \omega(x,y,t)\mathbf{k}$, where ω satisfies the two-dimensional vorticity-transport equation

$$\frac{\mathrm{D}\omega}{\mathrm{D}t} = \nu \boldsymbol{\nabla}^2 \omega.$$

Explain the physical significance of each term in this expression. Is vorticity conserved following the fluid in a two-dimensional viscous flow?

(ii) Show that there is a streamfunction $\psi(x,y,t)$ in terms of which the velocity $\mathbf{u}$, vorticity ω and two-dimensional vorticity-transport equation are given by

$$\mathbf{u} = \frac{\partial \psi}{\partial y}\mathbf{i} - \frac{\partial \psi}{\partial x}\mathbf{j}, \quad -\omega = \boldsymbol{\nabla}^2 \psi \equiv \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}, \quad \frac{\partial \omega}{\partial t} + \frac{\partial(\psi,\omega)}{\partial(y,x)} = \nu \boldsymbol{\nabla}^2 \omega.$$

(iii) By setting $\psi = \psi(r,\theta,t)$ in plane polar coordinates (r,θ), writing $\mathbf{u} = \boldsymbol{\nabla} \wedge (\psi \mathbf{k})$ and employing the chain rule

$$\frac{\partial(\psi,\omega)}{\partial(\theta,r)} = \frac{\partial(\psi,\omega)}{\partial(y,x)} \cdot \frac{\partial(y,x)}{\partial(\theta,r)},$$

show that the formulation in (c)(ii) becomes

$$\mathbf{u} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}\mathbf{e}_r - \frac{\partial \psi}{\partial r}\mathbf{e}_\theta, \quad -\omega = \boldsymbol{\nabla}^2 \psi \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2}, \quad \frac{\partial \omega}{\partial t} + \frac{1}{r}\frac{\partial(\psi,\omega)}{\partial(\theta,r)} = \nu \boldsymbol{\nabla}^2 \omega,$$

where $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are unit vectors in the r- and θ-directions.

Q6 The energy equation and dissipation.

(a) Assuming there is an external body force $\mathbf{F} = F_i \mathbf{e}_i$ acting per unit mass (*e.g.* gravity) and that there are no external energy sources (*e.g.* microwave heating), conservation of energy for a material volume $V(t)$ of a compressible conducting fluid is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\iiint_{V(t)} \rho c_v T \,\mathrm{d}V + \iiint_{V(t)} \frac{1}{2}\rho u_i^2 \,\mathrm{d}V\right) = \iint_{\partial V(t)} k\frac{\partial T}{\partial x_j} n_j \,\mathrm{d}S + \iiint_{V(t)} \rho u_i F_i \,\mathrm{d}V + \iint_{\partial V(t)} u_i \sigma_{ij} n_j \,\mathrm{d}S,$$

where T is the temperature, c_v the specific heat and k is the thermal conductivity. Explain the physical significance of each term in this expression.

(b) Assuming that c_v and k are constant, use the corollary to Reynolds' transport theorem and the divergence theorem to derive the energy equation in the form

$$\rho c_v \frac{\mathrm{D}T}{\mathrm{D}t} = k\boldsymbol{\nabla}^2 T + \Phi; \quad \Phi = \sigma_{ij}\frac{\partial u_i}{\partial x_j}.$$

For an incompressible Newtonian fluid, use the the symmetry of the rate-of-strain tensor e_{ij} to show that the dissipation is given by

$$\Phi = \frac{\mu}{2}\sum_{i,j=1}^{3}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2.$$

(c) An incompressible Newtonian fluid flows in an insulated container that occupies a region Ω. Use the energy equation to show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\Omega} \rho c_v T \,\mathrm{d}V = \iiint_{\Omega} \Phi \,\mathrm{d}V.$$

What is the physical significance of this expression?