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Viscous flow, Wed 20 May 10am

Can everyone see this?

2017 Q1 b, end

$$w = \alpha\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) + \beta$$

we need to find α and β, so that

$w = -U$ on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

and $\nabla^2 w = -\frac{1}{\mu} G$.

These cover:

2017 Q1

2016 Q1

2018 Q2

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2017, Q1 (iv) $w = \alpha\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) + \beta$

Flux $Q = \iint w \, dx dy$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1$, put $\begin{cases} x = ar\cos\theta \\ y = br\sin\theta \end{cases}$

$$Q = \int_0^{2\pi} d\theta \int_0^1 dr \, rab \, w$$

$$= \int_0^{2\pi} d\theta \int_0^1 dr \, rab \left(\alpha\left(r^2\cos^2\theta + r^2\sin^2\theta\right) + \beta\right)$$

$$= \int_0^{2\pi} d\theta \int_0^1 dr \, rab \left(\alpha r^2 + \beta\right)$$

$$= 2\pi ab \left[\frac{1}{4}\alpha r^4 + \frac{1}{2}\beta r^2\right]_0^1$$

$$= 2\pi ab \left(\frac{\alpha}{4} + \frac{\beta}{2}\right)$$

The flux $Q = 0$ when $\alpha = -2\beta$

Putting in $\alpha = \frac{Ga^2b^2}{2\mu(a^2+b^2)}$

$$\beta = -U - \frac{Ga^2b^2}{2\mu(a^2+b^2)}$$

from (iii) gives

$$U = -\frac{Ga^2b^2}{4\mu(a^2+b^2)}$$

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2016 Q1 (iii) Solve $\frac{\partial u}{\partial t} = \cos t + \frac{\partial^2 u}{\partial y^2}$

with $u = 0$ on $y = 0$, $u \to \sin t$ as $y \to \infty$

Put $u = \sin t + w(y,t)$

so $\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial y^2}$ with $w = -\sin t$ on $y = 0$

$w \to 0$ as $y \to \infty$

Try $w(y,t) = f(y)e^{it}$

and take real parts implicitly.

$\frac{\partial w}{\partial t} = i f(y) e^{it}$

$\frac{\partial^2 w}{\partial y^2} = f''(y) e^{it}$

We want $w = ie^{it} = -\sin t + i\cos t$ on $y = 0$

$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial y^2}$

$if = f''$ and $f(0) = i$

$f \to 0$ as $y \to \infty$

Try $f(y) = e^{\lambda y}$

$i = \lambda^2 = e^{i\pi/2}$

$\lambda = \pm e^{i\pi/4}$

For decay, $f \to 0$ as $y \to \infty$, we need $\mathrm{Re}\,\lambda < 0$, so $\lambda = -e^{i\pi/4} = -\frac{1+i}{\sqrt{2}}$

$\therefore \quad f(y) = f(0) e^{-\frac{1+i}{\sqrt{2}} y}$

$= i e^{-\frac{1+i}{\sqrt{2}} y}$

$w(y,t) = f(y) e^{it}$

$= i e^{it - \frac{1+i}{\sqrt{2}} y}$

$= e^{-y/\sqrt{2}}\, i e^{i(t - y/\sqrt{2})}$

The real part is

$w_{real} = e^{-y/\sqrt{2}} \sin\left(\frac{y}{\sqrt{2}} - t\right)$

Finally,

$u = \sin t + e^{-y/\sqrt{2}} \sin\left(\frac{y}{\sqrt{2}} - t\right)$

This does vanish on $y = 0$, and $u \to \sin t$ as $y \to +\infty$.

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2018 Q2 (c) $\quad F'' + 4F + \frac{F^2}{\varpi} + c = 0$

with $F = 0$ on $\theta = \pm\alpha$

$F(\theta) = F(0)\,\phi(\eta) \qquad \theta = \alpha\eta$

$F'' = F(0)\,\phi''(\eta)\left(\frac{d\eta}{d\theta}\right)^2 = \frac{F(0)}{\alpha^2}\phi''$

$(*)\quad \frac{F(0)}{\alpha^2}\phi'' + 4F(0)\phi + \frac{F(0)^2}{\varpi}\phi^2 + c = 0$

with BC $\phi = 0$ on $\eta = \pm 1$

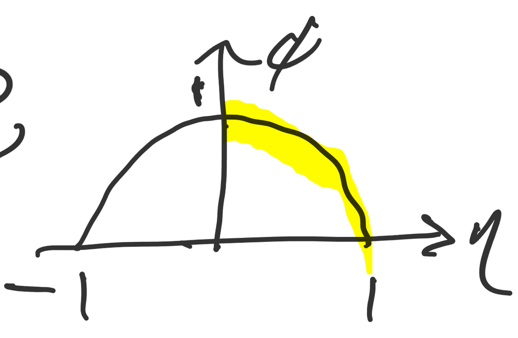
Structurally, this is $\phi'' = g(\phi)$

Multiply $(*)$ by ϕ' and $\int d\eta$

$\frac{F(0)}{\alpha^2}\phi''\phi' + 4F(0)\phi\phi' + \frac{F(0)^2}{\varpi}\phi^2\phi'$
$+ c = 0$

$\frac{F(0)}{2\alpha^2}\phi'^2 + \frac{4}{2}F(0)\phi^2 + \frac{F(0)^2}{3\varpi}\phi^3 + c\phi$
$= A$
$= \text{cst}$

The symmetric solution about $\eta = 0$ has $\phi'(0) = 0$, and we know $\phi(0) = 1$



$\therefore\ 2F(0) + \frac{F(0)^2}{3\varpi} + c = A$

$\frac{F(0)}{2\alpha^2}\phi'^2 + 2F(0)\phi^2 + \frac{F(0)^2}{3\varpi}\phi^3 + c\phi$

$= 2F(0) + \frac{F(0)^2}{3\varpi} + c$

$\frac{F(0)}{2\alpha^2}\phi'^2 = 2F(0)(1-\phi^2) + \frac{F(0)^2}{3\varpi}(1-\phi^3)$
$+ c(1-\phi)$

$\frac{1}{2\alpha^2}\phi'^2 = 2(1-\phi^2) + \frac{F(0)}{3\varpi}(1-\phi^3)$
$+ \frac{c}{F(0)}(1-\phi)$

$= (1-\phi)\left(\frac{c}{F(0)} + 2(1+\phi) + \frac{F(0)}{3\varpi}(1+\phi+\phi^2)\right)$

$= G(\phi)$

$\frac{1}{2\alpha^2}\left(\frac{d\phi}{d\eta}\right)^2 = G(\phi)$

$\frac{1}{\sqrt{2}\alpha}\frac{d\phi}{d\eta} = \pm\sqrt{G(\phi)}$

$\sqrt{2}\alpha\int d\eta = \pm\int\frac{d\phi}{\sqrt{G(\phi)}}$

$\sqrt{2}\alpha\eta = \mp\int_\phi^1\frac{ds}{\sqrt{G(s)}}$

$\mp$ as ϕ is in the lower limit

$\phi(\pm 1) = 0$ and $\phi(0) = 1$, so $\frac{d\phi}{d\eta} < 0$ in $\eta > 0$

$\therefore \sqrt{2}\alpha\eta = +\int_\phi^1\frac{ds}{\sqrt{G(s)}}$

$= \int_\phi^1\frac{ds}{\sqrt{(1-s)\left(\frac{c}{F(0)} + 2(1+s) + \frac{F(0)}{3\varpi}(1+s+s^2)\right)}}$

going to be part of K

The given answer has $R(1-s)s^2$ + lower powers of s.

Aiming for

$\alpha\eta = \int_\phi^1\frac{ds}{\sqrt{\frac{2}{3}R(1-s)\left(K + \left(1+\frac{6}{R}\right)s + s^2\right)}}$

$\Rightarrow \quad R = \frac{F(0)}{\varpi}$ and $K = 1 + 6\frac{\varpi}{F(0)} + \frac{3\varpi c}{F(0)^2}$

so K (or c) makes $\phi = 0$ when $\eta = 1$

You can also see Ian Hewitt's video version of this calculation. It's easier in hindsight to simplify the expression for $\left(\frac{d\phi}{d\eta}\right)^2$ to look like

$\left(\frac{d\phi}{d\eta}\right)^2 = \frac{2}{3}\alpha^2 R(1-\phi)\left(K + \left(1+\frac{6}{R}\right)\phi + \phi^2\right)$

before separating variables.