$U_s(x)\,\mathbf{i} \rightarrow$

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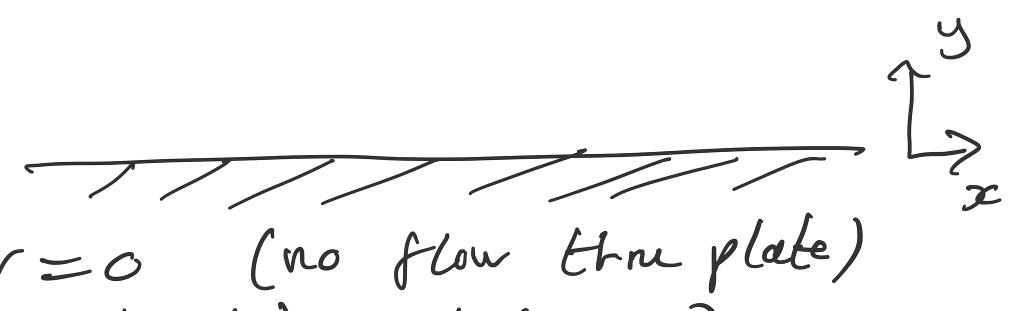
2016 Q2(a)



On $y=0$, $v=0$ (no flow thru plate)
$u = U_p(x) = b(x+c)$

Matching to the inviscid outer flow
$u \rightarrow U_s(x)$ as $Y \rightarrow \infty$

These are the 3 BC for the BL equations.

(b) Still the stretching plate:
$v=0$ and $u = b(x+c)$ on $Y=0$
$\psi \sim xy$ as $Y \rightarrow \infty$ for fixed x

$\Rightarrow u = \frac{\partial \psi}{\partial y} \rightarrow x$ as $Y \rightarrow \infty$ ''
i.e. $U_s(x) = x$.

BL scaling $y = Re^{-1/2} Y$
$v = Re^{-1/2} V$

$$\frac{\partial u}{\partial x} + \frac{\partial V}{\partial Y} = 0$$

$$u = \frac{\partial \Psi}{\partial Y} \text{ and } V = -\frac{\partial \Psi}{\partial x}$$

$V = -\frac{\partial \Psi}{\partial x} = 0$ on $Y=0$

$u = \frac{\partial \Psi}{\partial Y} = b(x+c)$ on $Y=0$

$u = \frac{\partial \Psi}{\partial Y} \rightarrow x$ as $Y \rightarrow \infty$

Try $\Psi = g(x) f(Y) + bch(Y)$

(+) $$\frac{\partial \Psi}{\partial Y}\frac{\partial^3 \Psi}{\partial Y^2 \partial x} - \frac{\partial \Psi}{\partial x}\frac{\partial^3 \Psi}{\partial Y^3} = \frac{\partial^4 \Psi}{\partial Y^4}$$

$$\frac{\partial}{\partial Y}\left(\frac{\partial \Psi}{\partial Y}\frac{\partial^2 \Psi}{\partial x \partial Y} - \frac{\partial \Psi}{\partial x}\frac{\partial^2 \Psi}{\partial Y^2}\right) = \frac{\partial^4 \Psi}{\partial Y^4}$$

Integrating in Y:
$$\frac{\partial \Psi}{\partial Y}\frac{\partial^2 \Psi}{\partial x \partial Y} - \frac{\partial \Psi}{\partial x}\frac{\partial^2 \Psi}{\partial Y^2} = \frac{\partial^3 \Psi}{\partial Y^3} + \text{fn}(x)$$

This is $-\frac{dp_0}{dx}$, the pressure gradient outside the BL

To get the expected solution with f'''', stick with (+) unintegrated

$$\frac{\partial \Psi}{\partial Y} = g(x) f' + bch'$$

$$\frac{\partial \Psi}{\partial x} = g' f$$

$$\left(g(x) f'(Y) + bch'(Y)\right)\left(g'(x) f''(Y)\right) - g' f\left(g(x) f''' + bch'''\right) = g(x) f''''(Y) + bch''''(Y)$$

$$g(x) g'(x) f' f'' + g'(x)\, bch' f'' - g' g\, f f''' - g' f\, bch''' = g f'''' + bch''''$$

Choose $g(x) = x$, $g'(x) = 1$

collecting terms proportional to x:
$$f' f'' - f f''' = f''''$$

collecting the remaining terms:
$$h' f'' - f h''' = h'''' \text{, put } H = h'$$

Boundary conditions:
$$\frac{\partial \Psi}{\partial x} = f(Y) = 0 \quad \text{on } Y=0$$

$$u = \frac{\partial \Psi}{\partial Y} = x f'(Y) + bch'(Y)$$

On $Y=0$, $u = b(x+c) = x f'(0) + bch'(0)$

$\Rightarrow f'(0) = b$, $h'(0) = 1$

As $Y \rightarrow \infty$, $u \rightarrow x$ so $f'(\infty) = 1$
$h'(\infty) = 0$

(ii) Equation for f:
$$f' f'' - f f''' = f''''$$
with $f(0) = 0$, $f'(0) = b$, $f'(Y) \rightarrow 1$ as $Y \rightarrow \infty$

This last condition $\Rightarrow$ $f'' \rightarrow 0$, $f''' \rightarrow 0$ as $Y \rightarrow \infty$

$$\left(f'^2 - f f''\right)' = f''''$$

LHS $= 2f' f'' - f' f'' - f f'''$ correct

Integrating w.r.t. Y
$$f'^2 - f f'' = f''' + \text{constant}$$

As $Y \rightarrow \infty$, $f'^2 \rightarrow 1$, $f'' \rightarrow 0$, $f''' = 0$

$\therefore$ constant $= 1$

b(iii) suppose $c = 0$ and $b = 1$
$u = b(x+c) = x$ on $Y=0$
$u \rightarrow x$ as $Y \rightarrow \infty$

The solution is just $u = x$ everywhere, so no BL.

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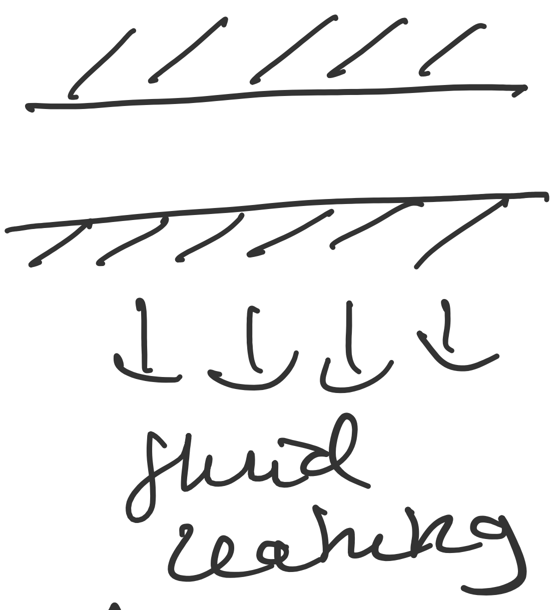
2014 Q3(c) Given σ_{ij}

The plates are horizontal



The upward force is (dimensionally)

$$\sigma_{33} = -p + 2\mu \frac{\partial w}{\partial z}$$

$$= -\Pi \hat{p} + 2\mu \frac{W}{\delta L} \frac{\partial \hat{w}}{\partial \hat{z}}$$

$$= -\underbrace{\frac{\mu W}{\delta^3 L} \hat{p}}_{O(1/\delta^2)\text{ larger}} + \frac{2\mu W}{\delta L} \frac{\partial \hat{w}}{\partial \hat{z}}$$

$O(1/\delta^2)$ larger, so just need this

In lubrication theory, the leading order normal force is just the pressure, so the upward force on the plate is

$$L \int_{-1}^{1} -\Pi \hat{p} (-1)\, d\hat{x}$$

normal n points down into fluid exerting the force on the plate above

$$= \frac{\mu W}{\delta^3} \int_{-1}^{1} \hat{p}\, d\hat{x}$$

$$= \frac{\mu W}{\delta^3} \int_{-1}^{1} \frac{48}{\pi^2} \cos \frac{\pi \hat{x}}{2}\, d\hat{x}$$

$$= \frac{192}{\pi^3} \frac{\mu W}{\delta^3}$$

on the upper plate

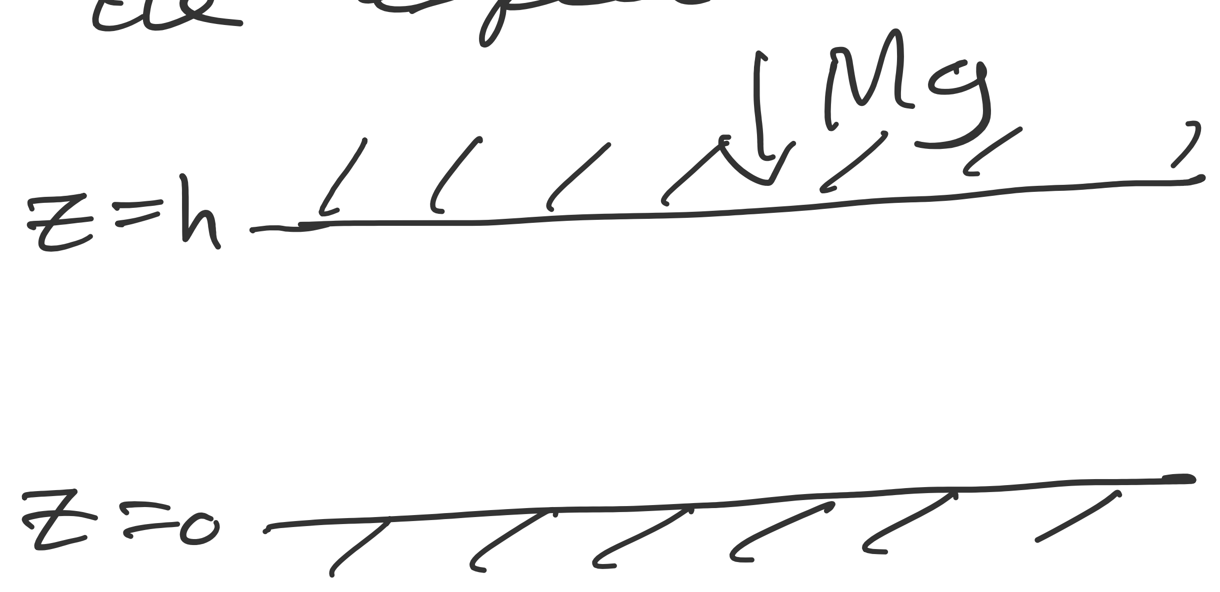
Instantaneous force balance, so

$$Mg = \frac{192}{\pi^3} \frac{\mu W}{\delta^3}$$

$\delta = h/L$ is the aspect ratio

$$\delta^3 = \frac{192 \mu W}{Mg \pi^3}$$

$$h = L \left(\frac{192 \mu W}{Mg \pi^3} \right)^{1/3}$$



Need $\delta \ll 1$ for this approach to be valid, so

$$\frac{192 \mu W}{Mg \pi^3} \ll 1, \qquad M \gg \frac{192 \mu W}{g \pi^3}$$

This question, the velocity scale W is prescribed, and the pressure scale $\Pi = \frac{\mu W}{\delta^3 L}$ balances the pressure gradient with the viscous force due to ∂zz.

The vertical momentum equation becomes $\frac{\partial \hat{p}}{\partial \hat{z}} = 0$.

For a thin film with a free surface flowing under gravity, the velocity scale $U = \frac{\delta^3 \rho g L^2}{\mu}$.

Choosing this instead of W, and U is a horizontal velocity not a vertical velocity scale, gives

$$\Pi = \frac{\mu U}{\delta^2 L} = \frac{\mu \delta^3 \rho g L^2}{\mu \delta^2 L}$$

$$= \rho g \delta L$$

$$\frac{\partial p}{\partial z} \sim \frac{\Pi}{\delta L} \sim \rho g \sim \text{gravitational force on the fluid}$$

We're ignoring gravity in the fluid in this question because $\rho g \ll \frac{\partial p}{\partial z}$ with these scalings.