Wednesday, 20 May 2020 18:53

2016 Q3 (b)

$u = 0$ on $r = 1$, no flux

$v = 1$ on $r = 1$, no slip on a moving boundary on the solid object

$$\frac{D}{Dt}(r - h(\theta, t)) = 0$$ on the free surface at $r - h(\theta, t) = 0$.

$$\Rightarrow \quad u = \frac{\partial h}{\partial t} + v \frac{\partial h}{\partial \theta}$$

Dynamic conditions on the free surface:

$\underline{\underline{\sigma}} \cdot \underline{n} = 0$ as there's a vacuum outside, so $\underline{\underline{\sigma}} = 0$ outside, and $\underline{\underline{\sigma}} \cdot \underline{n}$ is continuous at the free surface.

$$\underline{n} \cdot \underline{\underline{\sigma}} \cdot \underline{n} = 0 \quad \Rightarrow p = 0$$

$$\underline{t} \cdot \underline{\underline{\sigma}} \cdot \underline{n} = 0 \quad \Rightarrow \frac{\partial v}{\partial r} = 0$$

In (r, θ) coordinates

$$\underline{\underline{\sigma}} = \begin{pmatrix} -p + 2\mu \frac{\partial u}{\partial r} & \mu\left(\frac{\partial v}{\partial r} + \frac{\partial u}{\partial \theta}\right) \\ \mu\left(\frac{\partial v}{\partial r} + \frac{\partial u}{\partial \theta}\right) & -p + 2\mu \frac{\partial v}{\partial \theta} \end{pmatrix}$$

on the thin film approximation with dimensionless radius 1.

Wednesday, 27 May 2020 18:20

2016 Q2 (a)

Put $v = \delta_2 V(x, Y)$, $y = \delta_1 Y$

$$0 = \nabla \cdot \underline{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

$$= \frac{\partial u}{\partial x} + \frac{\delta_2}{\delta_1}\frac{\partial V}{\partial Y}$$

so choose $\delta_1 = \delta_2$.

$$u\frac{\partial u}{\partial x} + V\frac{\partial u}{\partial Y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{1}{\delta_1^2}\frac{\partial^2 u}{\partial Y^2}\right)$$

Choose $\delta_1 = Re^{-1/2}$ to get

$$u\frac{\partial u}{\partial x} + V\frac{\partial u}{\partial Y} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial Y^2} + \frac{1}{Re}\frac{\partial^2 u}{\partial x^2}$$

Expanding $u = u_0 + Re^{-1/2}u_1 + \cdots$

gives

$$u_0\frac{\partial u_0}{\partial x} + V_0\frac{\partial u_0}{\partial Y} = -\frac{\partial p_0}{\partial x} + \frac{\partial^2 u_0}{\partial Y^2}$$

at leading order.

Wednesday, 27 May 2020 18:39

2016 Q1(b)

We're told $\underline{u} = u(x,y,z,t)\underline{i}$

$$\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \nu\rho\nabla^2\underline{u}$$

$$\nabla\cdot\underline{u} = 0$$

$\nabla p = -\rho G\cos(\omega t)\underline{i}$, so

$$\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} = -G\cos(\omega t)\underline{i} + \nu\nabla^2\underline{u}$$

$$0 = \nabla\cdot\underline{u} = \frac{\partial u}{\partial x}$$

so $\underline{u} = u(y,z,t)\underline{i}$

$$\Rightarrow \underline{u}\cdot\nabla\underline{u} = 0$$

$$\frac{\partial \underline{u}}{\partial t} = G\cos(\omega t)\underline{i} + \nu\nabla^2\underline{u}$$

in the half-space $y > 0$.

The domain is independent of z, so we can take u to be independent of z, and find a solution for $u(y,t)$.

This is consistent with $\nabla\cdot\underline{u} = 0$ so we don't need an additional pressure gradient.

2016 Q1. 20th May 2020 morning session:

$$u(y,t) = \sin t + e^{-\frac{y}{\sqrt{2}}} \sin\left(\frac{y}{\sqrt{2}} - t\right)$$

The dimensionless stress on the plate is $\left.\frac{\partial u}{\partial y}\right|_{y=0}$.

$$\frac{\partial u}{\partial y} = e^{-\frac{y}{\sqrt{2}}}\left(-\sin\left(\frac{y}{\sqrt{2}} - t\right) + \cos\left(\frac{y}{\sqrt{2}} - t\right)\right)$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = (\sin t + \cos t)\frac{1}{\sqrt{2}}$$

$$= \sin(t + \pi/4)$$

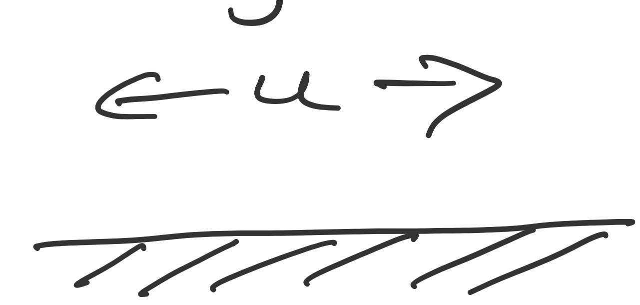
$$= \sqrt{2}\left(\sin t \cos\frac{\pi}{4} + \cos t \sin\frac{\pi}{4}\right)$$

as $\sin \pi/4 = \cos \pi/4 = \frac{1}{\sqrt{2}}$

$u \sim \sin t$ as $y \to \infty$

$\therefore$ $\left.\frac{\partial u}{\partial y}\right|_{y=0}$ is $\pi/4$ out of phase with $\lim_{y\to\infty} u(y,t)$.

bi) $$\frac{\partial u}{\partial t} = G\cos(\omega t) + \nu \frac{\partial^2 u}{\partial y^2}$$



$y=0$

The plate at $y=0$ is stationary, so $u=0$ on $y=0$.

Far from the plate ($y \gg 1$) we expect u to be independent of y, leaving $\frac{\partial u}{\partial t} = G\cos(\omega t)$

$$\Rightarrow \quad u \sim \frac{G}{\omega}\sin(\omega t).$$

Wednesday, 27 May 2020 18:27

Notes, page 20

$$\underline{t}(\underline{n}) = -p\underline{n} + \mu\left(2(\underline{n}\cdot\nabla)\underline{u} + \underline{n}\wedge(\nabla\wedge\underline{u})\right)$$

$$\underline{t}(n) = \underline{\underline{\sigma}}\cdot\underline{n}$$

and $\underline{\underline{\sigma}} = -p\underline{\underline{I}} + \mu\left((\nabla\underline{u}) + (\nabla\underline{u})^T\right)$

i.e.

$$\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$[\underline{t}(\underline{n})]_i = \sigma_{ij} n_j$$

$$= \left(-p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right) n_j$$

$$= -p n_i + \underline{\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) n_j}$$

$$[2(\underline{n}\cdot\nabla)\underline{u} + \underline{n}\wedge(\nabla\wedge\underline{u})]_i$$

$$= 2n_j\frac{\partial u_i}{\partial x_j} + \varepsilon_{ipq} n_p \varepsilon_{qrs}\frac{\partial u_s}{\partial x_r}$$

$$= 2n_j\frac{\partial u_i}{\partial x_j} + \varepsilon_{ipq}\varepsilon_{rsq} n_p\frac{\partial u_s}{\partial x_r}$$

$$= 2n_j\frac{\partial u_i}{\partial x_j} + (\delta_{ir}\delta_{ps} - \delta_{is}\delta_{pr}) n_p\frac{\partial u_s}{\partial x_r}$$

$$= 2n_j\frac{\partial u_i}{\partial x_j} + n_p\frac{\partial u_p}{\partial x_i} - n_p\frac{\partial u_i}{\partial x_p}$$

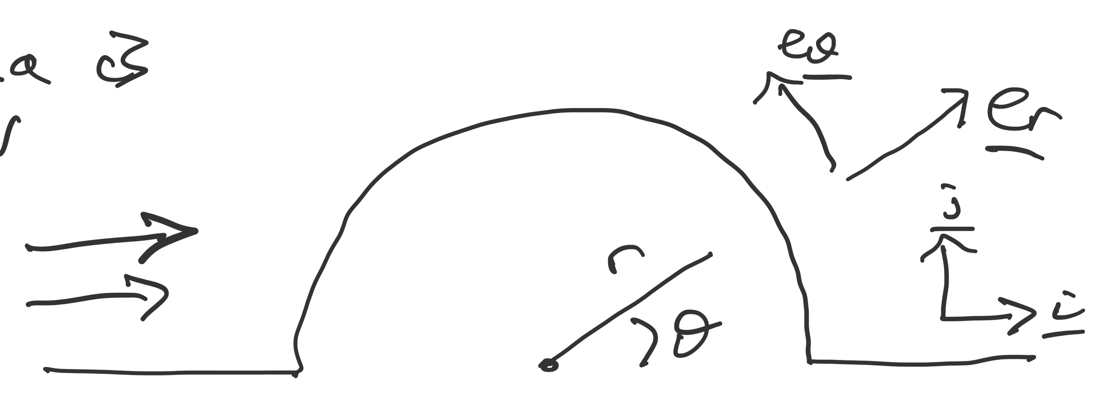
$$= 2n_j\frac{\partial u_i}{\partial x_j} + n_j\frac{\partial u_j}{\partial x_i} - n_j\frac{\partial u_i}{\partial x_j}$$

$$= \underline{n_j\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}$$ same as above

Wednesday, 27 May 2020 17:35

Example 2.6.4 in the notes (p.43)

Prandtl's idea is that the flow comprises an inviscid outer flow and a thin viscous boundary layer.



For inviscid flow around a cylinder:
The incoming uniform stream has no vorticity, and inviscid flows can't create vorticity, so

$$\underline{u} = \nabla\phi.$$

We need $\nabla\cdot\underline{u} = \nabla^2\phi = 0$, and

$$u_r = \frac{\partial\phi}{\partial r} = 0 \quad \text{on } r=1 \text{ (cylinder)}$$

$$u_x = u_r\cos\theta - u_\theta\sin\theta \to 1 \quad \text{as } r\to\infty$$

Let's try $\phi = f(r)\cos\theta$.

$$u_r = \frac{\partial\phi}{\partial r} = f'(r)\cos\theta$$

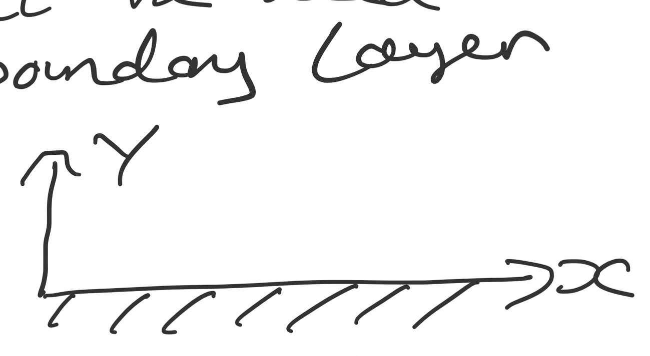
$$u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{f(r)}{r}\sin\theta$$

The solutions of Laplace's equation of the form $\phi = f(r)\cos\theta$ are $r\cos\theta$ or $\frac{1}{r}\cos\theta$.

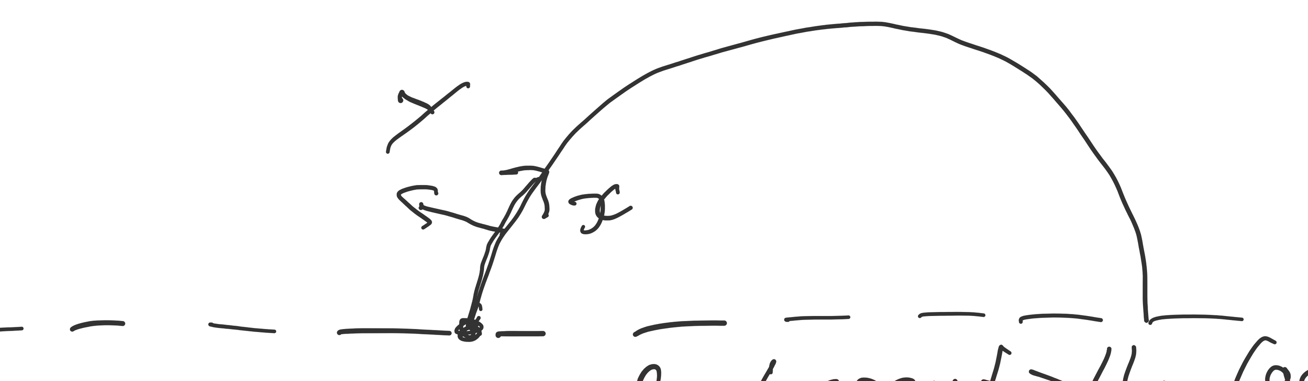
Imposing $\frac{\partial\phi}{\partial r} = 0$ on $r=1$ and $\phi \sim r\cos\theta = x$ as $r\to\infty$ determines $\phi = \left(r+\frac{1}{r}\right)\cos\theta$.

Now we know the outer flow, and hence U_s, but we need to put it onto boundary layer coordinates.

Originally we had



We can wrap this around a curved surface provided the radius of curvature is much bigger than the boundary layer thickness.



x is measured tangentially (arc length) along the boundary from the front, the forward stagnation point.
Y is measured normally from the boundary.

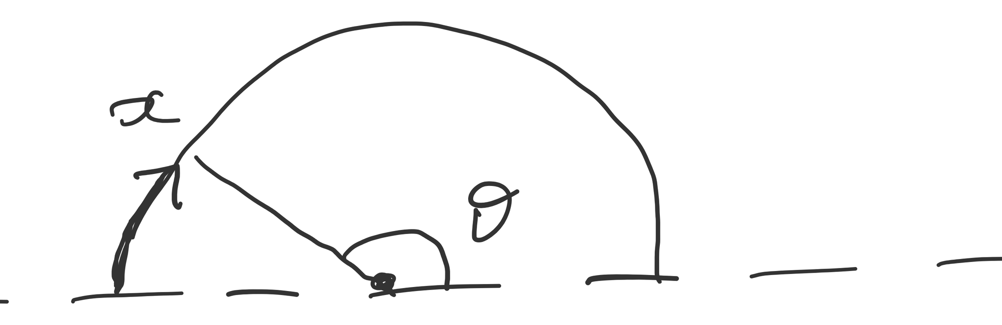
In the BL, $u(x,Y) \to U_s(x)$ as $Y\to\infty$, where

$$U_s(x) = \lim_{y\to 0} u_{outer}(x,y).$$

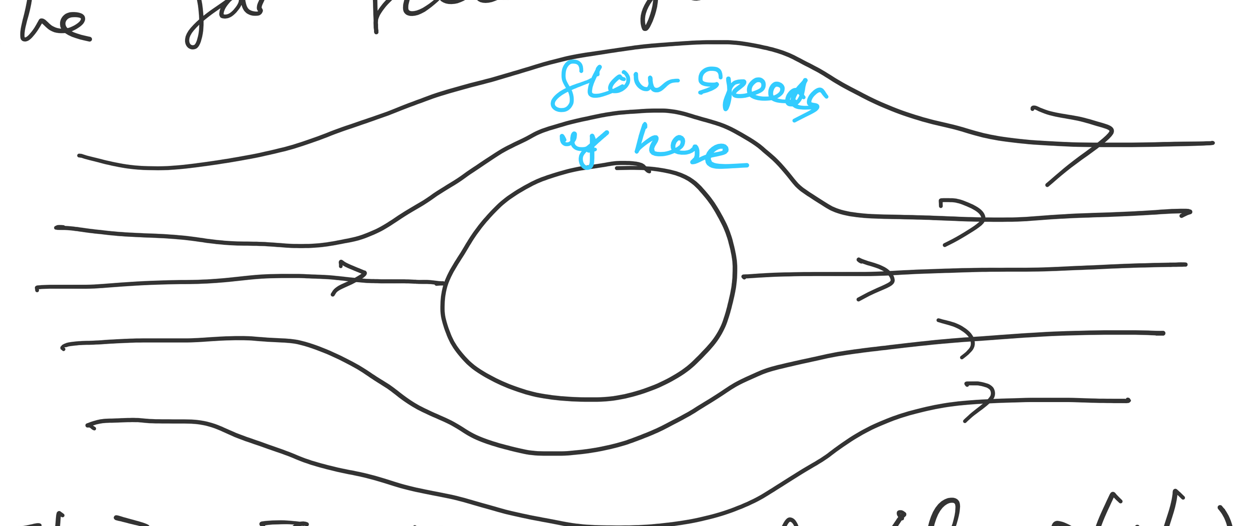
In other words,

$$\lim_{Y\to\infty} u_{BL}(x,Y) = \lim_{y\to 0} u_{outer}(x,y) = U_s(x).$$

$x = \pi - \theta$ as $r=1$



The far field flow looks like:

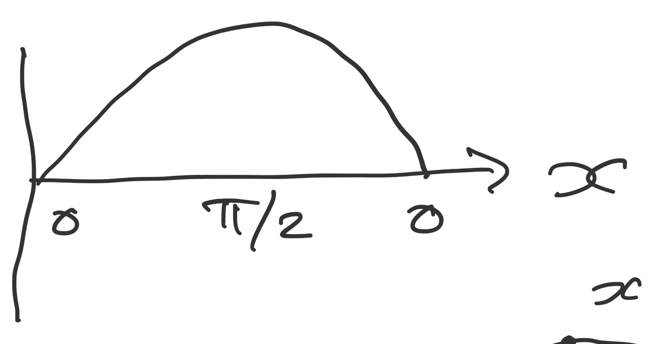


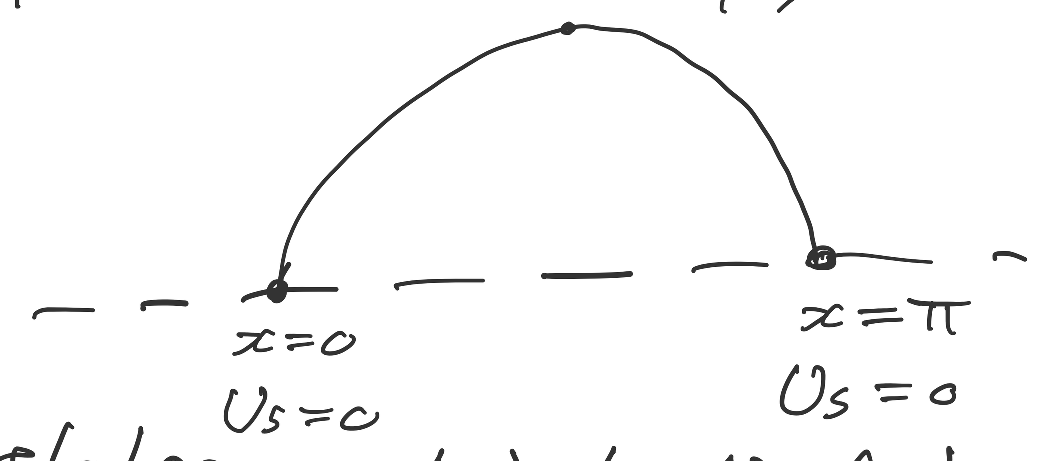
This is the far field relative to the thin boundary layer, but not relative to the cylinder.

$x = \pi - \theta$ on $r=1$

The tangential velocity in the inviscid outer flow as we approach the cylinder is

$$\lim_{y\to 0} -u_\theta(x,y) = -\frac{\partial\phi}{\partial\theta}(r=1, \theta=\pi-x)$$
$$= -(1+1)(-\sin(\pi-x))$$
$$= 2\sin x$$
$$= U_s(x)$$





Bernoulli relates $p_0(x)$ to $U_s(x)$ in the inviscid flow:

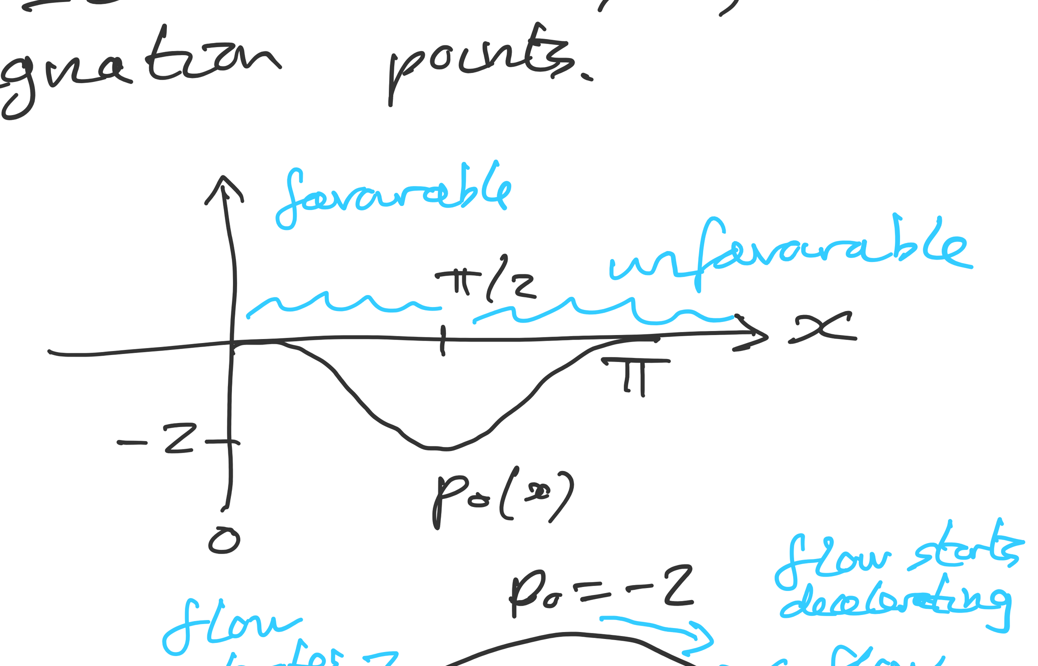
$$p_0(x) + \tfrac{1}{2}U_s^2(x) \quad \text{is constant.}$$

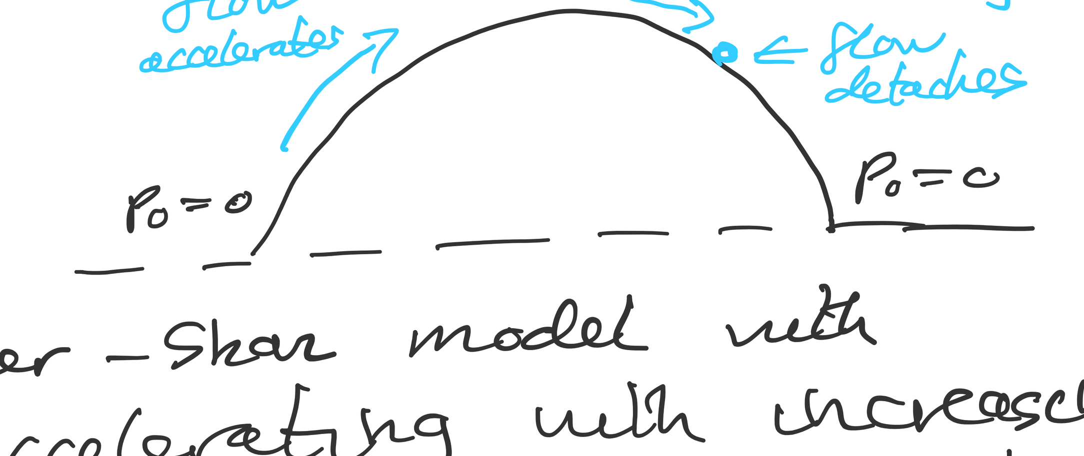
Differentiating w.r.t. x gives

$$\frac{dp_0}{dx} = -U_s\frac{dU_s}{dx} = -2\sin x\,(2\cos x)$$
$$= -2\sin(2x)$$

$$p_0(x) = \cos(2x) - 1$$

with the constant chosen to make $p_0 = 0$ at $x = 0, \pi$, the two stagnation points.





In the Falkner–Skan model with $U_s(x)$ is accelerating with increasing x (i.e. favourable pressure gradient) for $m>0$, and decelerating with increasing x for $m<0$.

Solutions to the Falkner–Skan equations exist for $m > -0.0904$ a little way into $m<0$.