B4.1 Functional Analysis I MT 2018: Problem Sheet 4

Except where indicated otherwise spaces may be assumed to be over $\mathbb{C}$.

1. Let c_0 be the space of all sequence which converge to zero, as usual equipped with the ∞-norm.

(i) Prove that $(c_0)^* = \ell_1$.

(ii) Let $S = \{s^{(n)}, n \in \mathbb{N}\}$ be given as the set of sequences defined by

$$s^{(n)} = (0, 0, \dots, 0, n, -(n+1), 0, \dots),$$

where the entry n occurs in the nth coordinate. Use (i) to determine whether $\operatorname{span}(S)$ is dense in c_0

(iii) If we regard S as a subset of the space ℓ^2 (equipped with its usual norm), is the $\operatorname{span}(S)$ dense in ℓ^2.

2. Let Y be a finite-dimensional subspace of a normed space X. Prove that there exists a continuous linear map $T : X \to Y$ so that $Ty = y$ for all $y \in Y$. Conclude that there exists a closed linear subspace Z such that $X = Y \oplus Z$.

Hint: It helps to introduce a basis of Y.

3. In this question assume the scalar field is $\mathbb{R}$.

(i) Consider $X = L^1[-1, 1]$ with the usual L^1 norm. Define $\varphi\colon L^1[-, 1] \to \mathbb{R}$ by

$$\varphi(f) = \int_0^1 f(t)\,\mathrm{d}t - \int_{-1}^0 f(t)\,\mathrm{d}t.$$

Show that $\varphi \in X^*$.

Show that $C := \{ f \mid \varphi(f) = 1 \}$ is a closed convex set containing infinitely many elements of minimum norm.

(ii) Assume X is a reflexive normed space, that is, every element of X^{**} is of the form $i(x)$ for some $x \in X$, where $i(x)(f) = f(x)$ for every $f \in X^*$. Prove that for each $f \in X^*$ there exists $x \in X$ such that

$$\|x\| = 1 \text{ and } f(x) = \|f\|.$$

(iii) Deduce that $C[-1, 1]$ is not reflexive.

4. Let X be a normed space and $T \in L(X)$. Let $T' \in L(X^*)$ be the associated dual operator. In (iii) and (iv) assume that X is a Banach space.

(i) Prove that for any $T \in L(X)$

$$\ker(T) = (T'X^*)_\circ \text{ and } \overline{TX} = \ker(T')_\circ$$

(ii) Prove that if T is invertible then T' is invertible and $(T')^{-1} = (T^{-1})'$.

(iii) Now assume T' is invertible. Prove that, for all $x \in X$,

$$\|Tx\| \geqslant \|(T')^{-1}\|^{-1}\|x\|.$$

[Hint: *You will need to make use of a consequence of HBT.*] Hence prove that T is invertible.

(iv) Prove that $\sigma(T) = \sigma(T')$.

In the following questions you may use all properties of the spectrum encountered in the lecture/lecture notes, in particular that the spectrum is non-empty, closed and contained in the closed disc around 0 *with radius* $\inf_{n\in\mathbb{N}} \|T^n\|^{1/n}$

5. A linear operator $T\colon \ell^1 \to \ell^1$ is defined by

$$T(x_1, x_2, x_3, \dots) = (y_1, y_2, y_3, \dots),$$
$$\text{where } y_k = \left(\frac{k+1}{k}\right) x_{k+1} \qquad \text{for } k \geqslant 1.$$

(i) Show that T is bounded and that $\|T\| = 2$. Obtain an explicit formula for T^2x and, more generally, for $T^n x$ when n is a positive integer and $x = (x_1, x_2, x_3, \dots) \in \ell^1$. Calculate $\|T^n\|$.

(ii) Which complex numbers λ are eigenvalues of T?

(iii) Prove that the spectrum of T is the disc $\{\, \lambda \in \mathbb{C} \mid |\lambda| \leqslant 1 \,\}$.

6. Let X be the space $C[0, 2]$ with the sup norm, and let

$$g(t) = \begin{cases} t & \text{if } 0 \leqslant t \leqslant 1, \\ 1 & \text{if } 1 < t \leqslant 2. \end{cases}$$

Define $T \in L(X)$ by

$$(Tf)(t) = g(t)f(t), \qquad f \in C[0, 2], \quad t \in [0, 2].$$

Find $\|T\|$, $\sigma_p(T)$ and $\sigma(T)$.

7. Let $T\colon \ell^\infty \to \ell^\infty$ be the right-shift operator given by $T(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots)$.

(i) Prove that $\sigma_{\mathrm{p}}(T) = \emptyset$.

(ii) Let $|\lambda| < 1$. Prove that $(\lambda I - T)$ does not map ℓ^∞ *onto* ℓ^∞.

(iii) Deduce that $\sigma(T) = \overline{\mathbb{D}}(0, 1)$,

Optional:

Part (iii) can alternatively be obtained by recognising T as the dual L' of the left-shift operator L on ℓ^1, for which the spectrum is discussed in the lecture notes. Can you also prove (i) and (ii) by making use of properties of dual operators?

8. Consider the operator $T\colon C[0, 1] \to C[0, 1]$ given by

$$(Tx)(t) = \int_0^t x(s)\,\mathrm{d}s \quad (t \in [0, 1]).$$

from Problem sheet 2. It is true and you may use that for every $n \in \mathbb{N}$ we have $T^n(x)(t) = \int_0^1 k_n(s, t)x(s)ds$ where $k_n(s, t) = \frac{(s-t)^{n-1}}{(n-1)!}$.

(i) Use this to show that $\sigma(T) = \{0\}$.
[Optional: Give an alternative, more direct proof by considering convergence of the series $\sum_{k=0}^\infty \lambda^{-k} T^k$.]

(ii) Let $S = (\mathrm{Id} + T)^{-1}$. Prove that $\sigma(S) = \{1\}$.