# B4.1: FUNCTIONAL ANALYSIS I

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This file contains the full set of webnotes for the course, with a contents list, by subsections, at the end.

*With acknowledgements to the lecturers who have previously shaped this course and who produced online notes for it, on which the present notes have drawn.*

Functional Analysis I studies **normed spaces** in general and **complete normed spaces** (called **Banach spaces**) in particular. Such spaces—principally infinite-dimensional ones—form the backbone of a theory that underpins much of applied analysis as well as being worthy of study in its own right.

The importance of normed spaces, in analysis and elsewhere in mathematics and its applications, is recognised by their introduction in the Part A Metric Spaces course.

The Part B course will ssume knowledge of the basic material from Part A. A summary is provided in Section 0 of these notes. The numbering is significant: Section 0 will not be covered in lectures.

## 0. Preliminaries and first examples, from Part A

*In which we set out the definition and simple properties of a norm, renew acquaintance with some familiar normed spaces, and review basic topological notions as these specialise to normed spaces.*

0.1. **Definitions: normed space, equivalent norms.**

Let $X$ be a vector space over $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A **norm** on $X$ is a function $x \mapsto \|x\|$ from $X$ to $[0, \infty)$ which satisfies, for all $x, y \in X$ and all $\lambda \in \mathbb{F}$,

(N1) $\|x\| \geqslant 0$ with equality if and only if $x = 0$;
(N2) $\|\lambda x\| = |\lambda| \|x\|$;
(N3) $\|x + y\| \leqslant \|x\| + \|y\|$.

We then say that $(X, \|\cdot\|)$ is a **normed space**. Where no ambiguity would result we simply say $X$ is a normed space. On the other hand we adopt the notation $\|x\|_X$ when we need to make the domain explicit.

Note that the restriction to real or complex scalars is needed in (N2). Later, when we work with two normed spaces at the same time—for example when considering maps from one space to another—we tacitly assume that $\mathbb{F}$ is the same for both.

We say that two norms, $\|\cdot\|$ and $\|\cdot\|'$ on a vector space $X$ are **equivalent** if there exist constants $m, M > 0$ such that, for all $x$,

$$m\|x\| \leqslant \|x\|' \leqslant M\|x\|.$$

0.2. **Basic properties of a norm and introductory examples.**

There's a (real or complex) vector space underlying every normed space $X$. So don't forget what you've learned in Linear Algebra courses in Prelims and Part A. It'll be the basics that come in handy in FA-I: the Subspace Test rather than Jordan Normal Form.

Obviously, $\mathbb{R}$ becomes a normed space when we take $\mathbb{F} = \mathbb{R}$ and $\|x\| = |x|$, and likewise for $\mathbb{C}$, with $\mathbb{F} = \mathbb{C}$. Indeed the norm conditions (N1) (non-degeneracy), (N2) (homogeneity) and (N3) (Triangle Inequality) mimic basic properties of modulus on $\mathbb{R}$ and on $\mathbb{C}$.

Some basic properties of modulus carry over to a general normed space $(X, \|\cdot\|)$ with the same proofs. We highlight

(i) $\|x_1 + \cdots + x_m\| \leqslant \|x_1\| + \cdots + \|x_m\|$ for any $m = 3, 4, \ldots$ and any $x_1, \ldots, x_m \in X$;
(ii) $\|x + y\| \geqslant \big|\|x\| - \|y\|\big|$ for all $x, y$ (**reverse triangle inequality**).

Care is needed in manipulating inequalities involving norms, just as it is in handling inequalities involving modulus.

In Section 1 we give a full discussion of what's involved in verifying the norm properties, working with a wide range of examples.

Those real or complex vector spaces which are equipped with an inner product carry a natural norm.

0.3. **Proposition (the norm on an inner product space).** *Let $\langle \cdot\,, \cdot \rangle$ be an inner product on a vector space $X$ (over $\mathbb{R}$ or $\mathbb{C}$). Then $\|x\| = \sqrt{\langle x, x\rangle}$ defines a norm on $X$.*

*Proof.* Note that $\|\cdot\|$ is well-defined because $\langle x, x\rangle \geqslant 0$ for all $x$. (N1) and (N2) follow directly from properties of the inner product. For (N3) we call on the Cauchy–Schwarz inequality:

$$|\langle x, y\rangle| \leqslant \|x\|\,\|y\|.$$

This gives

$$\begin{aligned}
\|x+y\|^2 &= \langle x+y, x+y\rangle \\
&= \|x\|^2 + \langle x, y\rangle + \langle y, x\rangle + \|y\|^2 \\
&= \|x\|^2 + 2\mathrm{Re}\langle x, y\rangle + \|y\|^2 \\
&\leqslant \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2 \\
&= (\|x\| + \|y\|)^2.
\end{aligned}$$

Since the norm is non-negative, $\|x+y\| \leqslant \|x\| + \|y\|$ follows. $\square$

0.4. **Subspaces.**

Part of the standard machinery of vector space theory involves the ability to form subspaces, We can consider a norm as an add-on to this general framework.

Given a normed space $(X, \|\cdot\|_X)$ and a subspace $Y$ of the vector space $X$, we can form a new normed space $(Y, \|\cdot\|_Y)$ by defining $\|y\|_Y = \|y\|_X$ for all $y \in Y$.

0.5. **Norms on the finite-dimensional spaces $\mathbb{F}^m$.** We can define the following norms on $\mathbb{F}^m$, for $\mathbb{F}$ as $\mathbb{R}$ or $\mathbb{C}$ and $m \geqslant 1$: for $x = (x_1, \ldots, x_m) \in \mathbb{F}^n$,

$$\begin{aligned}
\|x\|_2 &= \Big(\sum_{j=1}^m |x_j|^2\Big)^{1/2} \qquad (\textbf{Euclidean norm}) \\
\|x\|_1 &= \sum_{j=1}^m |x_j| \\
\|x\|_\infty &= \max_{j=1,\ldots,m} |x_j|.
\end{aligned}$$

Here the Euclidean norm comes from the standard inner product (the scalar product) and Proposition 0.3 confirms that it is indeed a norm. Verification that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ satisfy (N1), (N2) and (N3) is straightforward, using properties of the real and complex numbers.

These norms on $\mathbb{F}^m$ are related as follows:

$$\|x\|_2 \leqslant \|x\|_1 \leqslant \sqrt{m}\|x\|_2 \quad \text{and} \quad \frac{1}{\sqrt{m}}\|x\|_2 \leqslant \|x\|_\infty \leqslant \|x\|_2.$$

These inequalities tell us that any two of these norms are equivalent. In Section 1 we extend the definitions to define a norm $\|\cdot\|_p$ for any $p$ with $1 \leqslant p < \infty$.

0.6. **Further normed spaces encountered in Part A metric spaces.** The following appeared briefly as examples, with real scalars:

(1) The spaces $\ell^1$, $\ell^2$ and $\ell^\infty$. These are infinite-dimensional analogues of the finite-dimensional spaces with the analogous norms. Issues of convergence come into play here; see Section 1 for details.

(2) Function spaces
   - (i) Bounded real-valued functions on any set $\Omega$ with supremum norm: $\|f\|_\infty := \sup\{\,|f(x)| \mid x \in \Omega\,\}$. Boundedness ensures that $\|f\|_\infty$ is finite. Notation in FA-I is $\mathcal{F}^b(\Omega)$.

(ii) Real-valued continuous functions on a compact set $K$ with the supremum norm. Here boundedness is guaranteed. Notation: $C(K)$ or $C_{\mathbb{R}}(K)$. In particular, $K$ can be any closed bounded interval in $\mathbb{R}$.

(3) Continuous functions on a closed bounded interval with $L^1$ or $L^2$ norm.

These are just tasters. A more comprehensive catalogue of examples is given in Section 1.

0.7. **The metric associated with a norm.**

Let $(X, \|\cdot\|)$ be a normed space. Then $d\colon X \times X \to [0, \infty)$ defined by

$$d(x, y) = \|x - y\|$$

is a metric on $X$, so $\|x - y\|$ measures the **distance** between $x$ and $y$.

All the standard notions associated with a metric space are available in any normed space: open sets, closed sets, closure, and so on.

Suppose $X$ is a normed space with respect to two different norms, $\|\cdot\|$ and $\|\cdot\|'$. Then the norms are equivalent if and only if they give rise to the same open sets. This easy exercise is an extension of results proved in Part A.

Convergence of a sequence $(x_n)$ in a normed space $X$ is defined in the expected way. We say $(x_n)$ **converges** if there exists $x \in X$ such that $\|x_n - x\| \to 0$. In this situation we write $x_n \to x$. Note that convergence with respect to the supremum norm on a space of functions (as in 0.6) is *uniform convergence.*

Both closedness and continuity can, as in any metric space, be conveniently captured via sequences.

(i) A set $S$ is closed iff, for any sequence $(x_n)$,

$$x_n \in S \text{ for all } n \text{ and } x_n \to x \implies x \in S.$$

(ii) Given a non-empty subset $S$ of $X$, a point $x$ belongs to the closure $\overline{S}$ of $S$ iff there exists a sequence $(x_n)$ in $S$ such that $x_n \to x$.

(iii) A real- or complex-valued function $f$ on $X$ is continuous iff for all sequences $(x_n)$,

$$x_n \to x \implies f(x_n) \to f(x).$$

[Our reason for including (ii) here is to make it clear that there is no need to involve limit points, and no advantage in doing so.]

0.8. **Proposition.** *Let $X$ be a normed space. Then the following maps are continuous:*

(i) $x \mapsto \|x\|$, *from $X$ to $[0, \infty)$;*
(ii) $(x, y) \mapsto x + y$, *from $X \times X$ to $X$;*
(iii) $(\lambda, x) \mapsto \lambda x$, *from $\mathbb{F} \times X$ to $X$.*

*[Here the norm on $X \times X$ can be taken to be that given by $(x, y) \mapsto \|x\| + \|y\|$, or any norm equivalent to this.]*

*Proof.* See Problem sheet 0.

0.9. **Corollary (closure of a subspace).** *Let $Y$ be a subspace of a normed space $X$. Then the closure $\overline{Y}$ of $Y$ is also a subspace.*

0.10. **Advance notice: continuity of a linear map.**

It is natural in the context of normed spaces that we should consider maps between such spaces which are both linear and continuous. In Metric Spaces it was shown that these have a characterisation in terms of a condition of **boundedness**. The notion of a **bounded linear operator** and in particular of a **bounded linear functional** will be of major importance in FA-I (analysed in detail in Section 3 in the 2017 notes).

0.11. **Banach space: definition.**

Recall that a metric space $(X, d)$ is **complete** if every Cauchy sequence converges to an element of $X$. This notion applies in particular when the metric comes from a norm. A sequence $(x_n)$ in a normed space $(X, \|\cdot\|)$ is a **Cauchy sequence** if

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall m, n \geqslant N \quad \|x_n - x_m\| < \varepsilon.$$

A normed space $(X, \|\cdot\|)$ is a **Banach space** if it is complete, that is, every Cauchy sequence converges.

The following elementary result is very useful.

0.12. **Proposition (closed subspaces of Banach spaces).** *Let $(X, \|\cdot\|_X)$ be a normed space and $Y$ a subspace of $X$. Then*

(i) *If $(Y, \|\cdot\|_X)$ is a Banach space then $Y$ is closed in $X$.*
(ii) *If $(X, \|\cdot\|_X)$ is a Banach space and $Y$ is closed, then $(Y, \|\cdot\|_X)$ is a Banach space.*

*Proof.* This is just a definition-chase with sequences.

Consider (i). Suppose $Y$ is a Banach space and that $y_n \to x \in X$, where $(y_n)$ is a sequence in $Y$. Any convergent sequence is Cauchy and hence there exists $y \in Y$ such that $y_n \to y$. By uniqueness of limits, $x = y$, so $x \in Y$ and $Y$ is closed.

Consider (ii). Suppose $Y$ is closed and that $(y_n)$ is a Cauchy sequence in $Y$. Then $(y_n)$ is also a Cauchy sequence in the Banach space $X$. Hence there exists $x \in X$ such that $\|y_n - x\|_X \to 0$. Since $Y$ is closed, $x \in Y$. $\square$

We record the following Prelims result as a theorem. It underpins mathematical analysis at a deep level, and its implications are far-reaching. More parochially, we shall draw on it when showing that many normed spaces in our catalogue of examples here and in Section 1 are also Banach spaces.

0.13. **Theorem (Cauchy Convergence Principle).** *Each of $(\mathbb{R}, |\cdot|)$ and $(\mathbb{C}, |\cdot|)$ is a Banach space.*

# 1. Normed spaces, Banach spaces and Hilbert spaces

*In which we assemble our full cast of characters and start to get to know them.*

**Introductory course overview.**

**Context.**

All the spaces we shall consider will be real or complex normed spaces. Many of these will be important in analysis, both pure and applied. Infinite-dimensional spaces will predominate, with spaces of functions as primary examples. In rare but important cases the norm will come from an inner product.

Functional analysis as we study it involves **vector spaces** with additional structure (a norm function). Thus **linearity** is always present and all the maps we consider will be linear maps. This constrains the potential areas of application: mathematical physics yes; non-linear systems, no.

**The role of a norm.**

A norm on a real or complex vector space provides a special kind of metric, one which is compatible with the linear structure. In particular addition and scalar multiplication are continuous maps; see 0.8.

A norm provides a measure of distance. Different norms on the same vector space lead to different ways of measuring 'nearness', and the choice can be tailored to an intended application. So if we're dealing with, say, a space of differentiable functions we may want to use a norm which involves a function's derivative as well as the function itself; see 1.9.

As in $\varepsilon$-$\delta$ analysis the presence of a norm allows the study of **approximations**, with a measure of how good these are—an aspect of numerical analysis. But approximations are also a valuable theoretical tool: often any element of a normed space can be recognised as a limit (in norm) of elements drawn from some special subset. Here's a sample of such density results:

- the classic **Weierstrass's Theorem**, asserting that a continuous real-valued function on a closed bounded interval can be uniformly approximated by polynomials;
- the approximation of integrable functions by simple functions or by step functions, where the norm we use is derived from the integral.

The course contains numerous results of this general type, for a variety of spaces. Many are theorems of interest in their own right, others are principally useful in constructing proofs.

As Section 0 confirmed, we are able to make use of topological ideas, in a setting more restricted than that of general metric spaces. These ideas include open and closed sets; continuity; and completeness.

Many of the spaces we meet are **complete**, that is, Cauchy sequences converge. A complete normed space is a **Banach space**. A complete normed space whose norm comes from an inner product is called a **Hilbert space**. The latter spaces have special properties with a geometric flavour: think of them as behaving like Euclidean spaces.

**How prevalent is completeness?**

1. It turns out that ALL finite-dimensional normed spaces are complete, and all norms on a finite-dimensional space are equivalent. See Section 4, in which we fit finite-dimensional spaces into our overall theory.
2. Very many of our key infinite-dimensional examples are Banach spaces. Nevertheless normed spaces which are not complete are important in certain areas of application, notably PDE's. A source of these is non-closed subspaces of familiar Banach spaces. Only hints of this appear in FA-I.
3. The only concrete Hilbert spaces we see in FA-I are $\mathbb{F}^m$ with the Euclidean norm, the sequence space $\ell^2$ and $L^2$-spaces (the last with a light touch). (An in-depth study of Hilbert spaces, with further examples, forms a major part of FA-II.)

**The role of completeness in the theory**

Completeness does not feature strongly in the general **theorems** in FA-I. In particular the centrepiece **Hahn–Banach Theorem** is a theorem about normed spaces. Completeness is however not far away. The HBT concerns the **dual space** of a normed space $X$—defined as the space of continuous linear functionals on $X$—and this dual space is a Banach space. The HBT has multiple consequences: for example it allows us to analyse a normed space $X$ by taking 'snapshots': we look at how elements of its dual space act on it. Density will be a recurring theme in the course and the HBT leads to valuable density results.

Continuous linear maps feature prominently. Spectral theory is more complicated and more subtle in infinite-dimensional spaces than in the finite-dimensional ones with which Prelims LAII and Part A Linear Algebra worked. No Rank–Nullity Theorem here! Spectral theory in FA-I is explored in the context of Banach spaces (at the end of the course).

At the end of Section 2 we give an informal introduction to the special properties Hilbert spaces possess, thanks to their being Banach spaces with their norm coming from an inner product.

FA-I does not cover deep theory of Banach spaces. FA-II does venture into this. Any Banach space is a complete metric space, and such spaces have a rich theory thanks to the **Baire Category Theorem**. This has major applications to Banach spaces in general, with a clutch of Big Theorems, and to Hilbert spaces in particular, and FA-II includes an introduction to these. Moreover, many deep theorems in classical analysis stem from the BCT: the rich supply of continuous functions which are nowhere differentiable (it's most of them!); the existence of functions whose Fourier series behave atrociously in respect of pointwise convergence, . . . .

**How reliable is your intuition?**

You live in a Hilbert space world, $\mathbb{R}^3$, and draw sketch diagrams in the Hilbert space $\mathbb{R}^2$. You are accustomed to doing linear algebra in finite-dimensional spaces, backed up by the use of bases and by the Rank–Nullity Theorem. On the analysis side, the Heine–Borel Theorem characterises compact sets as those which are closed and bounded, and continuous real-valued functions on compact sets are bounded and attain their bounds.

In working with finite-dimensional spaces from an algebraic perspective, continuity and closure don't come into play. The theory of infinite-dimensional spaces is richer and more varied. We'd expect to include some topological assumptions: continuity of our linear maps; subspaces maybe needing to be closed. But the following are warnings that things may not always go smoothly.

(a) For a continuous linear map $T\colon X \to Y$, the kernel $\ker T$ is closed but the image $\operatorname{Im} T$ need not be. This explains why certain results in the finite-dimensional theory of dual spaces and dual maps require dimension arguments. Infinite-dimensional analogues are likely to bring in $\overline{\operatorname{Im} T}$.

In a Banach space setting useful sufficient conditions for $\operatorname{Im} T$ to be closed are available.

(b) Minimum distances need not be attained. Given a metric space $(X, d)$ and non-empty closed subset $S \subseteq X$ we can define the distance from $x$ to $S$ by

$$\operatorname{dist}(x, S) := \inf\{\, d(x, s) \mid s \in S \,\}.$$

It is an easy (Part A) topology exercise to show $x \mapsto \operatorname{dist}(x, S)$ is continuous. But even when $d$ is the metric coming from a norm on a Banach space and $S$ is a closed subspace, there may not be any $s_0 \in S$ such that $d(x, s_0) = \operatorname{dist}(x, S)$.

Replace 'Banach' by 'Hilbert' here and the result is true.

(c) Suppose we have a vector space direct sum $X = Y \oplus Z$ and consider the projection map $P_Y\colon y + z \mapsto y$ $(y \in Y,\ z \in Z)$. Any picture you are likely to draw will suggest that $\|P_Y(x)\| \leqslant \|x\|$. whence $P_Y$ is continuous. Don't be fooled!

Projection maps on general normed spaces need not be continuous (example on a problem sheet). Continuity is ensured if $X$ is a Banach space and $Y, Z$ closed (this fact rests on the BCT). Things work better still in a Hilbert space, for orthogonal projections.

Here (a), (b) and (c) reflect different extents to which the theory and practice of functional analysis in normed spaces is different from what you've seen hitherto. Divergence from the familiar is greatest in general normed spaces, less so in Banach spaces (though proofs may require hard work), and least in Hilbert spaces.

Now we embark in earnest on our study of normed spaces.

1.1. **Notes on verifying the norm properties.**

Suppose we have a vector space $X$ over $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, equipped with some function $x \mapsto \|x\|$ which we wish to show is a norm.

We have to confirm that $\|x\|$ is finite for every $x \in X$ (let's call his property (N0)) and that, for all $x, y \in X$ and all $\lambda \in \mathbb{F}$, the following hold:

(N1) $\|x\| \geqslant 0$ with equality if and only if $x = 0$;
(N2) $\|\lambda x\| = |\lambda|\|x\|$;
(N3) $\|x + y\| \leqslant \|x\| + \|y\|$.

For starters, we need to know $X$ really is a vector space. Virtually always $X$ will be, or can be identified with, a set of $\mathbb{F}$-valued functions on some set $\Omega$, and the addition and scalar multiplication will be given pointwise. Here the domain $\Omega$ doesn't have need to have any structure—just a set will do. The set $\mathbb{F}^\Omega$ of all functions from $\Omega$ to $\mathbb{F}$ is a vector space for pointwise addition and multiplication (from Prelims). So $X$ is a vector space so long as it's non-empty and closed under pointwise addition and scalar multiplication.

Now consider the norm conditions. Note that (N0), finiteness of $\|x\|$, may need checking. Where a space is specified as a subspace of a familiar vector space by looking at those elements for which the candidate norm is finite, then convergence questions may arise in the application of the Subspace Test; see 1.4 for an example.

The issue with (N1) is whether $\|x\| = 0$ implies $x = 0$. This may not be obvious (see Example 1.11(a) below) or even true (see Subsection 1.2).

Typically, (N2) is obvious or is boringly routine to check. One general observation can sometimes reduce the work. Assume $\|\mu x\| \leqslant |\mu|\|x\|$ holds for any $\mu \in \mathbb{F}$. Let $\lambda \neq 0$ and combine the inequalities we obtain by taking first $\mu = \lambda$ and then $\mu = 1/\lambda$ (some vector space axioms come into play here!). This gives $\|\lambda x\| = |\lambda|\|x\|$ for $\lambda \neq 0$; (N1) covers the missing case $\lambda = 0$.

The norm property which is most often non-trivial to check is the Triangle Inequality (N3). Note however the proof of (N3) that applies to the norm on an inner product space: the Cauchy–Schwarz inequality does the job, for all IPS's at once (recall the proof in 0.3).

1.2. **Quotients and seminorms.**

In the context of finite-dimensional vector spaces quotient spaces (as studied in Part A LA) are important as a tool for using induction on dimension to prove major results (triangularisation, Cayley–Hamilton Theorem, Spectral Theorem, for example). In this course we rarely encounter quotients of existing normed spaces.

When (N2) and (N3) hold but (N1) does not then we say that $\|\cdot\|$ is a **seminorm**. The game to play here is to pass to a quotient vector space to convert to a normed space. The process is set out in Problem sheet Q. 1. For concrete examples see 1.11(b) and (c).

We now bring on stage a full cast of characters for the FA-I course.

1.3. **More examples of norms on finite-dimensional spaces.**

We may define a norm on $\mathbb{F}^m$ for any $p \in [1, \infty)$ by

$$\|(x_1, \dots, x_m)\|_p = \left(\sum_{j=1}^{m} |x_j|\right)^{1/p}.$$

To confirm that the Triangle Inequality holds we can start from an inequality due to **Hölder** which reduces to the Cauchy–Schwarz inequality for $\mathbb{F}^m$ when $p = 2$. It states that for $x = (x_j)$, $y = (y_j) \in \mathbb{F}^m$ and $q$ such that $\frac{1}{p} + \frac{1}{q} = 1$,

$$\left|\sum_{j=1}^{m} x_j \overline{y_j}\right| \leqslant \left(\sum_{j=1}^{m} |x_j|^p\right)^{1/p} \left(\sum_{j=1}^{m} |y_j|^q\right)^{1/q}.$$

An optional exercise on Problem sheet 1 outlines a proof of this. From here one can go on to derive the Triangle Inequality for the $p$-norm on $\mathbb{F}^m$ (a version of **Minkowski's Inequality**), and well covered in textbooks.

1.4. **Sequence spaces.**

All the spaces $(\mathbb{F}^m, \|\cdot\|_p)$ $(1 \leqslant p \leqslant \infty)$ have infinite-dimensional analogues.

We first note that the set, denoted $\mathbb{F}^{\mathbb{N}}$, of infinite sequences $(x_j)_{j\geqslant 1}$, with coordinates $x_j \in \mathbb{F}$, form a vector space under the usual coordinatewise addition and scalar multiplication. (Here we tacitly assume that $\mathbb{N} = \{1, 2, \dots\}$; we shan't make a big deal of whether sequences should start with $x_1$ or $x_0$ and shall use whichever is appropriate in a given case.) Any subset of $\mathbb{F}^{\mathbb{N}}$ which satisfies the conditions of the Subspace Test is also a vector space.

We define

$$\ell^p = \{\, (x_j) \mid \sum |x_j|^p \text{ converges} \,\}, \qquad \|(x_j)\|_p = \left(\sum_{j=1}^{\infty} |x_j|^p\right)^{1/p} \qquad (1 \leqslant p < \infty)$$

and

$$\ell^\infty = \{\, (x_j) \mid (x_j) \text{ is bounded} \,\}, \qquad \|(x_j)\|_\infty = \sup |x_j|.$$

Very easy (AOL) arguments confirm that $\ell^\infty$ is a normed space. So now assume $1 \leqslant p < \infty$. We claim firstly that each $\ell^p$ is a vector space and secondly that $\|\cdot\|_p$ makes it into a normed space. The required arguments can be interwoven. This allows us to be both efficient and rigorous.

By way of illustration, we consider $\ell^1$. We set out the (Prelims-level) proof in some detail to show how to avoid being sloppy. [In particular we never write down an infinite sum $\sum_{j=1}^\infty a_j$ *before* we know that $\sum a_j$ converges.]

Let $(x_j)$ and $(y_j)$ be such that $\sum |x_j|$ and $\sum |y_j|$ converge. By the triangle inequality in $\mathbb{F}$,

$$|x_j + y_j| \leqslant |x_j| + |y_j| \text{ for all } j.$$

Hence, for all $n$,

$$s_n := \sum_{j=1}^n |x_j + y_j| \leqslant \sum_{j=1}^n |x_j| + \sum_{j=1}^n |y_j| \leqslant \|(x_j)\|_1 + \|(y_j)\|_1.$$

By the Monotonic Sequences Theorem applied to $(s_n)$, the series $\sum |x_j + y_j|$ converges and moreover

$$\|(x_j) + (y_j)\|_1 = \|(x_j + y_j)\|_1 \leqslant \|(x_j)\|_1 + \|(y_j)\|_1.$$

**Special case: $\ell^2$, as an IPS and as a normed space.**

You already know that $\mathbb{F}^m$ carries the usual Euclidean inner product (*alias* scalar or dot product) and that this provides an associated norm. No convergence issues here.

For $\ell^2$, convergence of the infinite sums which arise does have to be addressed. Probably the quickest route is to show that

$$((x_j), (y_j)) \mapsto \sum_j x_j \overline{y_j} \quad \text{for } (x_j), (y_j) \in \ell^2$$

is a well-defined inner product, and then to appeal to Proposition 0.3 for the norm properties. To this end, let $x = (x_j)$ and $y = (y_j)$ be in $\ell^2$. Then for any $n \geqslant 1$, using the CS inequality in the IPS $\mathbb{F}^n$,

$$\sum_{j=1}^n |x_j \overline{y_j}| \leqslant \left( \sum_{j=1}^n |x_j|^2 \right)^{1/2} \cdot \left( \sum_{j=1}^n |y_j|^2 \right)^{1/2} \leqslant \{x\| \cdot \|y\|;$$

We deduce that $\sum |x_j \overline{y_j}|$ converges. The inner product properties follow easily from those in the finite-dimensional case using arguments similar to those used to prove $\ell^1$ is a normed space.

**Remarks on the sequence spaces and their norms.**

All the $\ell^p$ norms are available on any finite-dimensional space, and all are equivalent. The situation is more complicated for the infinite-dimensional sequence spaces $\ell^p$. See Problem sheet Q. 5(i).

For the sequence spaces $\ell^p$, the choice $p = 2$, and no other, gives a norm coming from an inner product: The parallelogram law fails for all $p \neq 2$.

1.5. **Products of normed spaces.**

We can form the product of two vector spaces, by defining addition and scalar multiplication coordinatewise on the cartesian product of the underlying sets. Given normed spaces $(X_1, \|\cdot\|_{X_1})$ and $(X_2, \|\cdot\|_{X_2})$, we can define various norms on $X_1 \times X_2$:

$$(x_1, x_2) \mapsto \|x_1\|_{X_1} + \|x_2\|_{X_2} \quad \text{or} \quad (x_1, x_2) \mapsto \max\{\|x_1\|_{X_1}, \|x_2\|_{X_2}\},$$

for example. These mirror the 1-norm and $\infty$-norm on $\mathbb{F}^2$ (the case $X = Y = \mathbb{F}$). These two norms are equivalent, and either can be employed as convenient (or other choices, likewise).

1.6. **Subspaces of sequence spaces.**

Further examples of normed spaces arise as subspaces of familiar normed spaces. For example, the following are subspaces of $(\ell^\infty, \|\cdot\|_\infty)$ (bounded sequences):

- $c$: convergent sequences;
- $c_0$: sequences which converge to 0;
- $c_{00}$: sequences $(x_j)$ such that $x_j = 0$ for all but finitely many $j$.

Clearly $c_{00} \subsetneq c_0 \subsetneq c \subsetneq \ell^\infty$. Proofs that $c$, $c_0$ and $c_{00}$ are subspaces of $\ell^\infty$: use (AOL) from Prelims Analysis.

1.7. **Example: sum of subspaces.**

Here we present a cautionary tale: the vector space sum of two closed subspaces of a normed space need not be closed. There are various examples of this phenomenon, but all work in essentially the same way.

The example we give here lives in $\ell^2 \times \ell^1$ with $\|((x_j),(y_j))\| = \|(x_j)\|_2 + \|(y_j)\|_1$. Note that $\ell^1 \subseteq \ell^2$ (why?) and that $(1/j) \in \ell^2 \setminus \ell^1$. Let $Y$ and $Z$ be the subspaces given by

$$Y = \{\, (0, y) \mid y \in \ell^1 \,\} \quad \text{and} \quad Z = \{\, (x, x) \mid x \in \ell^1 \,\}.$$

Consider the sequence $\zeta_n = (1, 1/2, 1/3, \ldots, 1/n, 0, 0, \ldots)$. Then $(0, -\zeta_n) \in Y$ and $(\zeta_n, \zeta_n) \in Z$. This implies that $(\zeta_n, 0) \in Y + Z$. But $\zeta_n \to (1, 1/2, 1/3, \ldots) \in \ell^2$ so $(0, -\zeta_n)$ converges in $\ell^2 \times \ell^1$ but to an element which cannot belong to $Y + Z$. However $Y$ and $Z$ are closed.

[For a variation on the same theme see Problem sheet Q. 6.]

1.8. **Function spaces with the supremum norm.**

Frequent players on the Analysis stage: continuous functions, differentiable functions, infinitely differentiable functions, integrable functions, polynomials, ..., with appropriate domains, for example $\mathbb{R}$, a closed bounded interval $[a, b]$, $\mathbb{C}$. Without explicitly thinking about it, we perform vector space operations on classes of real- or complex-valued functions **pointwise**. Prelims Analysis confirms that, for example, $C[0, 1]$ is a vector space and that the solutions of $y'' + y = 0$ form a subspace of the space of all real-valued functions $y$ on $\mathbb{R}$.

Suppose $X$ is a vector space of bounded real- or complex-valued functions on some set $\Omega$. Then

$$\|f\|_\infty = \sup\{\, |f(t)| \mid t \in \Omega \,\}$$

defines a norm on $X$, known as the **sup norm**. We need to assume the functions are *bounded* to guarantee that $\|f\|_\infty$ is finite.

The following are important examples of spaces of $\mathbb{F}$-valued functions (with $\mathbb{F}$ as $\mathbb{R}$ or $\mathbb{C}$) which can carry the supremum norm:

- $\mathcal{F}^b(\Omega)$: all bounded functions on a set $\Omega$; the space $\ell^\infty$ is the case that $\Omega = \mathbb{N}$.

- $C[a,b]$, the continuous functions on a closed bounded interval $[a,b]$. More generally we can take $C(\Omega)$, for $\Omega$ any compact space. Here boundedness is guaranteed.
- $C^b(\mathbb{R})$, bounded continuous functions on $\mathbb{R}$.

1.9. **Example: norms on spaces of differentiable functions.**

The differentiable functions, or infinitely differentiable functions, on $[0,1]$ form subspaces of $C[0,1]$ (1-sided derivatives at the endpoints, as usual). These, and other examples in the same vein, are normed spaces for the sup norm. But it may be more appropriate, particularly in the context of approximations, to use a norm which better reflects the nature of the functions we would like to study. For example, $C^1[a,b]$ denotes the space of continuously differentiable functions on $[a,b]$ with norm given by

$$\|f\| = \|f\|_\infty + \|f'\|_\infty.$$

Certainly $\|f\|_\infty \leqslant \|f\|$, but the norms are not equivalent (recall Metric Spaces, sheet 1, Q. 4).

1.10. **Example: Lipschitz functions.**

See Problem sheet Q. 4. A case where some work is involved to show that the functions form a vector space and that the candidate norm really is a norm.

1.11. **Norms on spaces of integrable functions.**

When dealing with functions on $\mathbb{R}$ or on subintervals of $\mathbb{R}$ (or on some general measure space) we often want to allow some averaging when measuring how close together two functions are. Using integrals to measure distance can capture this.

(a) We start with a simple example, from Prelims and picked up again in Part A Metric Spaces. Consider $C[0,1]$ and define

$$\|f\|_1 = \int_0^1 |f(t)|\,\mathrm{d}t.$$

It follows from elementary properties of integrals that $\|f\|_1$ is finite and $\geqslant 0$, and that (N2) and (N3) hold. It is a Prelims result, too, that

$$\int_0^1 |f(t)|\,\mathrm{d}t = 0 \implies f \equiv 0$$

(this relies on continuity of $f$: argue by contradiction, recalling that $|f(c)| > 0$ forces $|f(t)| > 0$ in some interval $[0,1] \cap (c-\delta, c+\delta)$ ...). A detailed proof can be found in Metric Spaces notes. Hence $(C[0,1], \|\cdot\|_1)$ is a normed space.

(b) **The Lebesgue spaces** [presupposing Part A Integration]
Consider

$$\mathcal{L}^1(\mathbb{R}) = \{\, f\colon \mathbb{R} \to \mathbb{R} \mid f \text{ is measurable and } \textstyle\int |f| < \infty \,\}.$$

This is a vector space under the usual pointwise operations. The map $f \mapsto \int |f|$ cannot be a norm because $\int |f| = 0$ implies $f = 0$ almost everywhere (by an MCT application) but $f$ need not be the zero function. We have a seminorm but not a norm.

We then let

$$\mathcal{N} = \{\, f\colon \mathbb{R} \to \mathbb{R} \mid f = 0 \text{ a.e.} \,\}.$$

This is a subspace of $\mathcal{L}^1(\mathbb{R})$ and we can form the quotient space $L^1(\mathbb{R}) = \mathcal{L}^1(\mathbb{R})/\mathcal{N}$, whose elements are the equivalence classes for the relation $\sim$ given by $f \sim g$ iff

$f = g$ a.e.. If we denote the equivalence class of $f$ by $[f]$ we get a well-defined norm on $L^1(\mathbb{R})$ by setting

$$\|[f]\| = \int |f|.$$

This is an instance of the general procedure for obtaining a normed space from a space which carries a seminorm $\|\cdot\|_0$ by quotienting by the subspace $\{\, x \mid \|x\|_0 = 0 \,\}$, as set out in Problem sheet Q.1. In practice we allow some sloppiness and speak about elements of $L^1(\mathbb{R})$ are if they were ordinary functions but do not distinguish between functions which are equal a.e. and usually drop the [ ] notation.

It is worth noticing that each equivalence class $[f]$ can contain at most one function which is continuous everywhere. So, for example, we don't get into difficulties if we treat $C[0,1]$ as if it were a subspace of $L^1[0,1]$.

Similarly, for $p \in (1,\infty)$ we define $L^p(\mathbb{R}) = \mathcal{L}^p(\mathbb{R})/\mathcal{N}$ where

$$\mathcal{L}^p(\mathbb{R}) = \{\, f \colon \mathbb{R} \to \mathbb{R} \mid f \text{ is measurable and } \textstyle\int |f|^p < \infty \,\}.$$

Once again, the case $p = 2$ is special. The $L^2$-norm comes from an inner product: for the complex case,

$$\langle f, g\rangle := \int f\overline{g}.$$

The fact that the traditional norm on an $L^p$ space indeed gives a normed space will be assumed in FA-I. A proof of the Triangle Inequality appeared in Part A Integration, starting from **Hardy's inequality**. These results are important and detailed accounts are available in many textbooks.

(c) For completeness we record that $L^\infty(\mathbb{R})$ is defined to be the space of (equivalence classes of) bounded measurable functions $f \colon \mathbb{R} \to \mathbb{F}$, with

$$\|f\|_\infty = \inf\{\, M > 0 \mid |f(t)| \leqslant M \text{ a.e.} \,\}.$$

[This space is mentioned for only for completeness.]

The Lebesgue spaces are much more central to FA-II than to FA-I. In this course they feature primarily as illustrations. In FA-II $L^2$-spaces, associated with various measure spaces, are central. The $\ell^p$-spaces, for $1 \leqslant p < \infty$ can be subsumed within the $L^p$-spaces using counting measure.

**Technical note.** Where spaces of integrable functions arise in the course as examples or, very occasionally, on problem sheets, measurability of the functions involved may be assumed. FA-I is not a course on measure theory. Non-measurable functions are anyway elusive beasts, and are not encountered in everyday mathematics; their existence relies on an assumption from Set Theory (Zorn's Lemma).

To conclude this section we record some of the special properties shared by IPS-based examples in which $\|x\| = (\langle x, x\rangle)^{1/2}$. We collect together, for occasional use later, the basic toolkit for working with such a norm.

1.12. **Properties of the norm on an inner product space.**

(i) **Cauchy–Schwarz inequality**: Let $x, y \in X$. Then

$$|\langle x, y\rangle| \leqslant \|x\|\,\|y\|,$$

with equality if and only if $x$ and $y$ are linearly dependent.

(ii) **Parallelogram Law**:

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

A norm (on a real or complex vector space) for which the Parallelogram Law fails cannot come from an inner product (see Problem 0.2 for an example).

[**Aside.** There is a converse. Let $\|\cdot\|$ be a norm on a vector space $X$ such that the Parallelogram Law holds. Then there is an inner product $\langle\cdot,\cdot\rangle$ on $X$ such that $\|x\| = \sqrt{\langle x, x\rangle}$. This is a bit tricky to prove.]

(iii) **Polarisation**: Retrieving the inner product from the norm:

$$\langle x, y\rangle = \begin{cases} \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right) & (\mathbb{F} = \mathbb{R}), \\ \frac{1}{4}\left(\|x+y\|^2 + \mathrm{i}\|x+\mathrm{i}y\|^2 - \|x-y\|^2 - \mathrm{i}\|x-\mathrm{i}y\|^2\right) & (\mathbb{F} = \mathbb{C}). \end{cases}$$

Proof: expand out the expressions on the RHS.

(iv) $\langle\cdot,\cdot\rangle$ is a continuous function from $X \times X$ to $\mathbb{F}$. Proof: Exercise (note Proposition 0.8).

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## 2. Completeness and density

*In which we establish the completeness of many of the spaces assembled in Section 1 and non-completeness of a few others, and present some general techniques.*

With our cast of examples now assembled on stage, we start to explore their individual characters against a backdrop of general properties. Here the topology of normed spaces provides the focus.

### 2.1. Density.

Recall that a subset $S$ of a metric space (or more generally a topological space) $X$ is **dense** if $\overline{S} = X$. Density is an important notion: it will enable us to talk about approximations of general elements by special ones, and many proofs in functional analysis and its applications proceed by establishing a given result first for elements of a dense subset, or subspace, of a normed space and then extending the result to all of $X$ by a limiting argument.

First examples of dense subspaces:

(a) Let $X = \ell^p$ where $1 \leqslant p < \infty$). Let $e_n = (\delta_{jn})$, that is, the sequence all of whose coordinates are 0 except the $n^{\text{th}}$ coordinate, which is 1. Let $x = (x_j)$ be an arbitrary element of $X$. Then

$$\|x - \sum_{k=1}^{n} x_k e_k\|_p = \|(0, 0, \ldots, 0, x_{k+1}, \ldots)\|_p \to 0 \text{ as } k \to \infty.$$

Hence *the subspace of $\ell^p$ consisting of vectors having at most finitely many non-zero coordinates is dense in $\ell^p$.*

(b) [Assumed fact] The step functions are dense in $L^p(\mathbb{R})$ for $1 \leqslant p < \infty$. A corresponding result holds for $L^p(a, b)$, where $-\infty \leqslant a < b \leqslant \infty$.

In functional analysis, step function approximations are usually much easier to work with than approximations by simple functions.

### 2.2. Density and examples of non-completeness.

The contrapositive of Proposition 0.12(i) leads to easy, and natural, examples of normed spaces which fail to be complete. Suppose $Y$ is a dense and proper subspace of a normed space $X$. Then $\overline{Y} = X$ so $Y$ cannot be closed and so cannot be complete for the norm inherited from $X$.

**Example:** Consider $C[-1,1]$ with norm $\|f\|_1 = \int_{-1}^{1} f(t)\,\mathrm{d}t$. To prove non-completeness, consider the sequence $(f_n)$ of continuous piecewise-linear functions for which

$$f_n(t) = \begin{cases} -1 & \text{if } -1 \leqslant t \leqslant -1/n, \\ nt & \text{if } -1/n < t < 1/n, \\ 1 & \text{if } 1/n \leqslant t \leqslant 1. \end{cases}$$

Suppose for contradiction $\|f_n - g\|_1 \to -$ where $g \in C[-1.1]$. But $\|f_n - f\|_1 \to 0$, where $f = \chi_{)0,1]} - \chi_{[-1,0)}$. This implies $g = f$ a.e., which is not possible.

We are ready for a clutch of archetypal completeness proofs. All involve function spaces. There's some overlap with Part A Metric Spaces but we give several proofs here to reinforce the key points in the strategy. Direct proofs via Cauchy sequences follow a uniform pattern. Subsidiary results can obtained via "closed subspace of a Banach space is a Banach space" (from 0.12).

2.3. **Example: completeness of $\mathcal{F}^b(\Omega)$ ($\ell^\infty$ is a special case).**

We take $X$ to be the space of real-valued bounded functions on a set $\Omega$ with the sup norm. Let $(f_n)$ be a Cauchy sequence in $X$. Then given $\varepsilon > 0$, there exists $N$ such that

$$\forall m, n \geqslant N \quad \sup_{s \in \Omega} |f_m(s) - f_n(s)| < \varepsilon.$$

**Step 1: identify candidate limit.** Fix $t \in \Omega$, and let $m, n \in \mathbb{N}$. Then

$$|f_m(t) - f_n(t)| \leqslant \sup_{s \in \Omega} |f_m(s) - f_n(s)| = \|f_m - f_n\|_\infty.$$

It follows that $(f_n(t))$ is a Cauchy sequence in $\mathbb{R}$. Hence there exists a real number, write it as $f(t)$, such that $f_n(t) \to f(t)$.

**Step 2: from pointwise convergence to sup norm convergence (they're different!).** For $s \in \Omega$,

$$m, n \geqslant N \implies |f_m(s) - f_n(s)| < \varepsilon.$$

With $s$ and $n$ fixed let $m \to \infty$ to get

$$n \geqslant N \implies |f(s) - f_n(s)| \leqslant \varepsilon.$$

It follows that $\|f - f_n\|_\infty \to 0$ as $n \to \infty$.

**Step 3: membership condition (that is, $f \in X$).** We need $f$ bounded. Using the inequality in Step 2 and the Triangle Inequality we get, for all $s$,

$$|f(s)| \leqslant |f_N(s)| + \varepsilon \leqslant \|f_N\|_\infty + \varepsilon.$$

Hence $\|f\|_\infty \leqslant \|f_N\|_\infty + \varepsilon < \infty$. □

2.4. **Example: completeness of $C[0,1]$.**

We just outline the steps, noting that the framework is as in 2.3. Let $(f_n)$ be a Cauchy sequence in $X = C[0,1]$.

**Step 1: candidate limit.** $(f_n)$ Cauchy implies for each $t$ that $(f_n(t))$ is Cauchy in $\mathbb{R}$, and hence convergent, to $f(t)$ say, for each $t$.

**Step 2: From pointwise convergence to sup norm convergence**. As in 2.3.

**Step 3: $f \in X$.** The issue is continuity of $f$. But what we need is exactly that the uniform limit of continuous functions is continuous—true by the standard "$\varepsilon/3$ argument" from Prelims.

An alternative here is simply to show that $C[0,1]$ is a closed subspace of the Banach space $\mathcal{F}^b([0,1])$. This amounts to piggybacking on Example 2.3 to reduce the problem to Step 3 alone.

### 2.5. Example: completeness of $\ell^1$.

The issue here is as much with the notation as with the substance. We start from a Cauchy sequence in $\ell^1$, which we shall denote by $(x^{(n)})$, where $x^{(n)} = (x_j^{(n)})$.

**Step 1: candidate limit, coordinatewise.** Since $|x_j^{(m)} - x_j^{(n)}| \leqslant \|x^{(m)} - x^{(n)}\|_1$ so for each $j$ the sequence of $j^{\text{th}}$ coordinates is Cauchy and so converges, to some $x_j$. Let $x = (x_j)$.

**Step 2: convergence of $(x_j^{(n)})$ to $(x_j)$ in $\ell^1$-norm.** For any $K \in \mathbb{N}$ we have

$$\sum_{j=1}^{K} |x_j^{(m)} - x_j^{(n)}| \leqslant \|x^{(m)} - x^{(n)}\|_1$$

and the RHS can be made less than a given $\varepsilon$ if $m, n \geqslant N$, for some $N \in \mathbb{N}$. Keeping $K$ and $n$ fixed and letting $m \to \infty$, (AOL) gives

$$\sum_{j=1}^{K} |x_j - x_j^{(n)}| \leqslant \varepsilon \quad \text{for all } n \geqslant N.$$

Since this is true for all $K$ we deduce that $(x_j - x_j^{(n)})$ belongs to $\ell^1$ for all $n \geqslant N$ and that its norm tends to 0 as $n \to \infty$.

**Step 3: $x = (x_j) \in \ell^1$.** This follows from Step 2 and the triangle inequality for $\ell^1$, by writing $x$ as $(x_j - x_j^{(N)}) + (x_j^{(N)})$.

A similar but messier proof shows that $\ell^p$ is complete for $1 < p < \infty$.

### 2.6. Example: completeness of $c_0$.

An exercise in $\varepsilon$-$\delta$ analysis.—be careful with quantifiers! Show that $(c_0, \|\cdot\|_\infty)$ is a closed subspace of $\ell^\infty$ and so complete.

### 2.7. Example: completeness of $C^1[0,1]$.

Consider the space $X$ of real-valued continuously differentiable functions on $[0,1]$ with the $C^1$ norm: $\|f\| = \|f\|_\infty + \|f'\|_\infty$, as in 1.9. Let $(f_n)$ be a Cauchy sequence in $X$.

Because $\|f_m - f_n\| \geqslant \|f_m - f_n\|_\infty$ and $\|f_m - f_n\| \geqslant \|f_m' - f_n'\|_\infty$ we can show as in Example 2.4 that there exist continuous functions $f$ and $g$ such that $\|f - f_n\|_\infty \to 0$ and $\|g - f_n'\|_\infty \to 0$. It remains to show that $f' = g$. Once we have this, $\|f - f_n\| \to 0$ follows easily by (AOL).

Observe that (by FTC, from Prelims Analysis III),

$$f_n(t) - f_n(0) = \int_0^t f_n'(s)\,\mathrm{d}s \quad \text{(for all } n\text{)}.$$

Let $n \to \infty$ to get

$$f(t) - f(0) = \int_0^t g(s)\,\mathrm{d}s;$$

here we use uniform convergence of $(f_n')$ to justify interchanging the limit and the integral on the RHS. But, again from Analysis III, the indefinite integral $\int_0^t g(s)\,\mathrm{d}s$ is differentiable w.r.t. $t$ with derivative $g(t)$. We conclude that $f' = g$, as required.

2.8. **Stocktake on tactics for completeness proofs.**

Note that all the completeness proofs presented so far for function spaces (which subsume sequence spaces) follow the same pattern. Suppose we have such a space $X$, which we wish to show is complete. We take a Cauchy sequence $(x_n)$ in $X$ and first identify a candidate limit, $x$ say, a point at a time (or a coordinate at a time).

We then need to show two things: that $x \in X$ and that $x_n \to x$ *with respect to* $X$*'s norm*. For this we usually show that $x \in X$ by showing that $x - x_N$ is in $X$ for suitable $N$ and deducing that $x = (x - x_N) + x_N$ is also in $X$. The proof that $x - x_N \in X$ usually comes out of showing that $\|x - x_n\|$ is small for $n \geqslant N$ some sufficiently large $N$. This strategy underlies our ordering of Steps 2 and 3 in our examples above.

Note that Step 1, identifying the limit, pointwise, does not make Step 2 unnecessary. On $C[0,1]$ for example, convergence w.r.t. the sup norm is uniform convergence, a much stronger condition than pointwise convergence.

A significant consequence of the Cauchy Convergence Principle in elementary Analysis is that every absolutely convergent series $\sum a_n$ of real or complex numbers is convergent. The proof relies on consideration of the sequences of partial sums of $\sum |a_n|$ and $\sum a_n$. We now carry these ideas over to normed spaces and Banach spaces.

2.9. **Series of vectors in a normed space.**

Let $(x_n)$ be a sequence in a normed space $X$ and consider the sequence $(s_n)$ of partial sums:

$$s_n = x_1 + \cdots + x_n \quad (n \geqslant 1).$$

If there exists $x \in X$ such that $s_n \to x$ then we write $x = \sum_{n=1}^{\infty} x_n$ and say that the series $\sum x_n$ **converges**. The series is said to be **absolutely convergent** if $\sum \|x_n\|$ converges.

2.10. **Theorem (completeness and absolute convergence).** *Let $X$ be a normed space. Then $X$ is a Banach space if and only if every absolutely convergent series of vectors in $X$ converges.*

*Proof.* $\Longrightarrow$ : this is proved in exactly the same way as for the case in which $X$ is $\mathbb{R}$ or $\mathbb{C}$, with $\|\cdot\|$ in place of $|\cdot|$—check Prelims notes!

$\Longleftarrow$: Take any Cauchy sequence $(y_n)$ in $X$. We endeavour to construct a series $\sum x_n$ in $X$ which is absolutely convergent and such that $x := \sum_{n=1}^{\infty} x_n$ supplies the limit we require for $(y_n)$. We can find natural numbers $n_1 < n_2 < \cdots$ such that

$$\|y_\ell - y_m\| < 2^{-k} \quad \text{for } \ell, m \geqslant n_k.$$

Let $s_k = y_{n_k}$. Define $(x_k)$ by

$$x_1 = s_1, \qquad x_k = s_k - s_{k-1} \quad (k > 1).$$

Then the real series $\sum \|x_k\|$ converges by comparison with $\sum 2^{-k}$ so $\sum x_k$ is absolutely convergent, and hence by assumption it converges. Moreover, by construction, $\sum x_k$ is a telescoping series:

$$y_{n_k} = s_k = \sum_{\ell=1}^{k} x_\ell.$$

We deduce that $(y_{n_k})$ converges. But, just as in Prelims Analysis, a Cauchy sequence in a normed space which has a convergent subsequence must itself converge. $\square$

2.11. **Applications of Theorem 2.10.**

(a) **Completeness of the $L^p$ spaces $(1 \leqslant p < \infty)$** (Riesz–Fischer Theorem). The proof requires mastery of techniques of Lebesgue integration and won't be discussed in FA-I.

A spin-off is completeness of the $\ell^p$ spaces: consider counting measure on $\mathbb{N}$.

(b) **Completeness of $L^\infty(\mathbb{R})$.** [Outline proof included just for completeness.] Take an absolutely convergent series $\sum f_n$ in $L^\infty(\mathbb{R})$. By definition of the norm on $L^\infty(\mathbb{R})$, there exists for each $n$ a null set $A_n$ such that $|f_n(t)| \leqslant \|f_n\|_{L^\infty(\mathbb{R})} + 2^{-n}$. Let $A = \bigcup A_n$; this is null. Now let $g_n = f_n \chi_{\mathbb{R}\setminus A}$. Then each $g_n \in \mathcal{F}^b(\mathbb{R})$ and $\sum \|g_n\|_{\mathcal{F}^b(\mathbb{R})}$ converges by comparison since $\sum(\|f_n\|_{L^\infty} + 2^{-n})$ converges.

By completeness of $\mathcal{F}^b(\mathbb{R})$ (see 2.3), $\sum g_n$ converges, to some bounded function $G$. Moreover, off $A$, $\sum_{k=1}^n f_k = \sum_{k=1}^n g_k \to G$ pointwise. Therefore $G$ is measurable, $G \in L^\infty(\mathbb{R})$ and (for convergence in norm),

$$\Big\|G - \sum\nolimits_{k=1}^n g_k\Big\|_{\mathcal{F}^b(\mathbb{R})} \geqslant \Big\|G - \sum\nolimits_{k=1}^n f_k\Big\|_{L^\infty} \to 0 \text{ as } n \to \infty.$$

(c) **Quotient spaces**. It can be proved from Theorem 2.10 that if $X$ is a Banach space and $Y$ is a closed subspace of $X$ then $X/Y$ is a Banach space for the quotient norm:

$$\|x + Y\| := \inf\{\, \|x + y\| \mid y \in Y \,\}.$$

(Here closedness of $Y$ is necessary to ensure that we have a norm rather than merely a seminorm.)

We shan't need this result in FA-I. The proof is rather technical and we omit it.

We conclude this section with some indications of why Hilbert spaces are special and what distinguishes them from inner product spaces in general and from Banach spaces in general. This brief account can be seen as providing context to FA-I and as a look-ahead to FA-II. [*The theorems mentioned below do not form part of the examinable syllabus for FA-I*].

2.12. **A glimpse at Hilbert spaces.** The Prelims and Part A Linear Algebra courses reveal features and methods that apply only to inner product spaces. Euclidean spaces, along with much of their geometry, form the prototype for finite-dimensional IPS's. A central notion is that of **orthogonality** and a key theorem asserts that if $L$ is a subspace of a fd IPS $V$ then

$$V = L \oplus L^\perp. \tag{†}$$

This is a key step in the proof of the Spectral Theorem for self-adjoint operations in finite-dimensional IPS's, since it allows to proceed by induction on $\dim V$.

This can be proved using ideas of the Gram–Schmidt process and orthonormal bases. More geometrically, given $x \in V$, one want $y \in L$ so that $x - y \in L^\perp$, which then implies $V = L + L^\perp$. *Geometrically*, we want to pick $y_x \in L$ so $d(x, y_x)$ is as small as possible. Algebraic arguments tell us this will work when the IPS $V$ is finite-dimensional. But does it work in general?

We want

$$d(x, y_x) = \delta := \inf\{\, d(x, y) \mid y \in L \,\},$$

that is, $\|x - y_0\| = \inf\{\, \|x - y\| \mid y \in L \,\}$. A viable strategy, **in a Hilbert space** is to take a sequence $(y_n)$ in $L$ such that $\|x - y_n\| \to \delta$ and to apply the parallelogram law to show $(y_n)$ is Cauchy, and so convergent. Finally we'd need to add the assumption that $L$ is **closed** to get $y_x := \lim y_n \in L$. By continuity of the norm function, $y_0$ is the closest point we seek.

Hence:–

**Theorem** (**Closest Point Theorem, for subspaces**) *Let $L$ be a closed subspace of a Hilbert space $X$. Given $x \in L$ there exists a unique point $y_x \in L$ such that*
$$d(x, y_x) = \inf\{\, d(x, y) \mid y \in L \,\}.$$

This leads on to a core theorem about Hilbert spaces. (n Section 4 we'll see that any subspace of a finite-dimensional space is closed, so the theorem subsumes the result for finite-dimensional IPS's recalled above. Concerning the final assertion of the theorem recall the comments about projections in the preamble to Section 1.

**Theorem** (Projection Theorem) *Let $X$ be a Hilbert space and $L$ a closed subspace of $X$. Then $X = L \oplus L^{\perp}$.*

*Moreover $\|P_L(x)\| \leqslant \|x\|$ for all $x \in X$, where $P_L$ is the projection map from $X$ onto $L$.*

Full details and the derivation from the Closest Point Theorem can be found in textbooks covering basic Hilbert space theory.

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## 3. Linear operators between normed spaces

*In which we bring on stage structure-preserving maps between normed spaces (the* **bounded linear operators***) and develop their elementary properties.*

It is taken for granted in contemporary pure mathematics that we should consider not just mathematical objects having a particular type (groups, vector spaces, metric spaces, ...) but also the structure-preserving maps between such objects (respectively, homomorphisms, linear maps, continuous maps, ...). That is, we study objects not in isolation but we study how they relate to other objects of the same type. Furthermore, this approach ties in with potential applications: for example, on a space of infinitely differentiable functions, the properties of differential operators may be important in the mathematical modelling of physical problems (though we do not treat these aspects in this course).

In Part A Metric Spaces an indication was given that a map from one normed space to another has special properties when it is linear AND continuous. We shall call such a map a **continuous linear operator**. We now elaborate on what was shown in Metric Spaces. A map $T\colon X \to Y$ satisfying condition (3) in Proposition 3.1 will be said to be **bounded**.

3.1. **Proposition (characterising continuous linear operators).** *Let $X$ and $Y$ be normed spaces over $\mathbb{F}$ and assume that $T\colon X \to Y$ is linear. Then the following conditions are equivalent:*
1. *$T$ is continuous;*
2. *$T$ is continuous at $0$;*
3. *there exists a non-negative constant $M$ such that $\|Tx\| \leqslant M\|x\|$ for all $x \in X$.*

*Proof.* (1) $\Longrightarrow$ (2): Trivial.

(2) $\Longrightarrow$ (3): Suppose for contradiction that (3) fails. Then we can find a sequence $(x_n)$ in $X$ such that $\|x_n\| \leqslant 1$ and $\|Tx_n\| > n$ (for all $n$). Let $y_n = x_n/n$ so that $y_n \to 0$ by (N2). By (2), $Ty_n \to T0$. But $T$ linear and (N2) together imply $\|Ty_n\| > 1$ and $T0 = 0$, which gives the required contradiction.

(3) $\Longrightarrow$ (1): Let $x_0, x \in X$ Then

$$\|Tx - Tx_0\| = \|T(x - x_0)\| \leqslant M\|x - x_0\|.$$

Hence $T$ is continuous (using the $\varepsilon$–$\delta$ definition of continuity of a map between metric spaces). $\square$

### 3.2. **The norm of a bounded linear operator.**

Again let $X$ and $Y$ be normed spaces over $\mathbb{F}$. Write $\mathcal{B}(X,Y)$ for the set of all continuous, *alias* bounded, linear operators from $X$ to $Y$, and $\mathcal{B}(X)$ for $\mathcal{B}(X,X)$.

The set $\mathcal{B}(X,Y)$ becomes a vector space if we define addition and scalar multiplication *pointwise* in the usual way: given $S, T \in \mathcal{B}(X,Y)$ and $\lambda \in \mathbb{F}$, we "define" $S+T$ and $\lambda T$ by

$$\forall x \in X \quad (S+T)(x) = Sx + Tx,$$
$$\forall x \in X \quad (\lambda T)(x) = \lambda Tx.$$

As in Prelims LA, $S+T$ and $\lambda T$ are linear maps. Moreover it is easy to see from Proposition 3.1 that they are continuous as maps from one normed space to another. Check the details!

We'd like to do more: we'd like to make $\mathcal{B}(X,Y)$ into a normed space in its own right. Given $T \in \mathcal{B}(X,Y)$ it is easy to check that the following candidate definitions for $\|T\|$ all give the same value:

(Op1) $\|T\| = \sup\left\{ \dfrac{\|Tx\|}{\|x\|} \mid x \neq 0 \right\}$;

(Op2) $\|T\| = \sup\{ \|Tx\| \mid \|x\| = 1 \}$;

(Op3) $\|T\| = \sup\{ \|Tx\| \mid \|x\| \leqslant 1 \}$;

(Op4) $\|T\| = \inf\{ M \mid \forall x\big(\|Tx\| \leqslant M\|x\|\big) \}$.

Here (Op1), (Op2) and (OP3) are minor variants of each other and we use them interchangeably, as convenient; the link to (Op4) relies on the definition of a sup as the *least* upper bound. We leave the proofs of equivalence as an exercise, noting as usual that $\|x/\|x\|\|$ has norm 1 *provided* $x \neq 0$. Don't forget about the special case $x = 0$.

It is a routine exercise to verify that $T \mapsto \|T\|$ does define a norm on $\mathcal{B}(X,Y)$. Note also that (0p1) immediately supplies the very useful fact that

$$\|Tx\| \leqslant \|T\|\,\|x\| \quad \text{for all } x \in X.$$

### 3.3. **Bounded linear operators and their norms: first examples.**

Suppose that $X, Y$ are normed spaces over $\mathbb{F}$ and that you are asked to show that some given map $T$ belongs to $\mathcal{B}(X,Y)$. You need to check

(i) $T$ is a well-defined map from $X$ to $Y$. Usually this just means you have to confirm that, for each $x \in X$, the formula for $Tx$ makes sense and that $Tx \in Y$. (For example, are there convergence issues to be addressed?)
(ii) $T$ is a linear map. This is likely to be routine checking and should not be lingered over.
(iii) $T$ is bounded. Usually this can be done by estimating $\|Tx\|$ to exhibit a finite constant $M$ such that $\|Tx\| \leqslant M\|x\|$ for all $x$.

If you are also asked to calculate $\|T\|$, (iii) will have told you that $\|T\| \leqslant M$. You then have to find the minimum possible $M$. See the examples for illustrations of how to do this.

(1) **Shift operators on sequence spaces.** To illustrate, we work with maps from $\ell^1$ to $\ell^1$. Other examples are available. Define

$$R(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots),$$
$$L(x_1, x_2, \ldots) = (x_2, x_3, \ldots).$$

it is clear that each of $R$ and $L$ maps $\ell^1$ into $\ell^1$ and is linear. Also, for any $x = (x_j) \in \ell^1$,

$$\|Rx\| = \|x\| \text{ and } \|Lx\| \leqslant \|x\|.$$

Hence $R, L \in \mathcal{B}(\ell^1, \ell^1)$, with $\|R\| = 1$ immediate from (Op3) and $\|L\| \leqslant 1$.

Let $e_n = (\delta_{jn})_{j\geqslant 1}$. Then $\|e_n\| = 1$ for all $n$. Then $\|Le_2\| = \|e_1\| = 1$ and we deduce that $\|L\| = 1$, with the supremum in (Op3) being attained.

(2) Let $X = Y = C_{\mathbb{R}}[0,1]$ and define $Tf$ by $(Tx)(t) = tx(t)$ (for all $t \in [0,1]$). We claim that $T \in \mathcal{B}(X)$ and $\|T\| = 1$. Prelims Analysis confirms that $T$ maps $X$ into $X$ and is linear [no proof called for here]. Also, for all $x \in X$ and $t \in [0,1]$,

$$|(Tx)(t)| = |tx(t)| \leqslant |x(t)| \leqslant \|x\|_\infty.$$

Hence $\|Tx\| \leqslant \|x\|$ for all $x \in X$ and we have equality when $x \equiv 1$. It follows that $T$ is bounded, with $\|T\| = 1$.

(3) $T$ as in (2) but now $X = Y = L^2(0,1)$. Other $L^p$ spaces are available.

First note that $g\colon t \mapsto t^2$ is a bounded measurable function on $[0,1]$, so $x \in L^2(0,1)$ implies $Tx \in L^2(0,1)$ too and linearity of $T$ is clear. Also

$$\|Tx\|_2^2 = \int_0^1 |tx(t)|^2 \,\mathrm{d}t \leqslant \int_0^1 |x(t)|^2 \,\mathrm{d}t = \|x\|_2^2.$$

Therefore $T$ is bounded and $\|T\| \leqslant 1$.

It is *not* immediately obvious that $\|T\| \geqslant 1$: we can't claim that the sup defining $\|T\|$ in (Op1) or (Op2) will be attained. To prove that the sup really is 1, we construct a 'witnessing sequence'. Define $x_n$ by

$$x_n(t) = \sqrt{n}\chi_{[1-\frac{1}{n},1]}(t) \quad (t \in [0,1]).$$

Routine calculation gives $\|x_n\|_2^2 = \sqrt{n}^2/n = 1$ and

$$\|Tx_n\|_2^2 = \int_0^1 |tx_n(t)|^2 \,\mathrm{d}t = n\int_{1-1/n}^1 t^2 \,\mathrm{d}t \geqslant n(1-1/n)^2/n = (1-1/n)^2.$$

Hence $\|T\| \geqslant (1-1/n)$ for all $n$ so $\|T\| \geqslant 1$.

[Observe that the supremum cannot be shown to be attained because members $x$ of $X$ are not necessarily such that $x(t) = \mathrm{O}(t^2)$ as $t \to 1$.]

(4) **An integral operator.** Let $X = C[0,1]$, real-valued continuous functions on $[0,1]$ with the sup norm. Let $k \in C([0,1]\times[0,1])$. Note that, as a continuous real-valued function on a compact set, $k$ is bounded. For $x \in X$ let

$$(Tx)(t) = \textstyle\int_0^1 k(s,t)x(s)\,\mathrm{d}s \quad (t \in [0,1]).$$

Note that the integral on the RHS exists for each $t$ since the integrand is continuous. We claim that $Tx \in C[0,1]$ for each $x \in X$. The slickest way to prove this is probably to use the Continuous DCT to show

$$\lim_{t\to t_0} \textstyle\int_0^1 k(s,t)x(s)\,\mathrm{d}s = \int_0^1 k(s,t_o)\,\mathrm{d}s :$$

a dominating function for the family of functions $y_t\colon s \mapsto k(s,t)x(t)$ on $[0,1]$ is $\|k\|_\infty\|x\|_\infty\chi_{[0,1]}$. [Alternative strategies: (i) use 'ordinary' DCT for sequences and appeal to 2.1 or (ii) use elementary estimation of $|(Tx)(t) - (Tx)(t_0)|$, making use of the fact that $k$ is uniformly continuous.] Hence $T\colon C[0,1] \to C[0,1]$ and by elementary properties of integrals $T$ is linear. Moreover, for all $t$,

$$|(Tx)(t)| = \left|\int_0^1 k(s,t)x(s)\,\mathrm{d}s\right| \leqslant \int_0^1 |k(s,t)x(s)|\,\mathrm{d}s \leqslant \|k\|_\infty \|x\|_\infty.$$

Hence $\|Tx\|_\infty \leqslant \|k\|_\infty \|x\|_\infty$ and therefore $T$ is bounded, with $\|T\| \leqslant \|k\|_\infty$.

See Problem sheet Q. 8 for a special case in which $\|T\|$ can be calculated explicitly.

3.4. **Example: an unbounded operator.**

Let $X$ be the space of real-valued continuously differentiable functions on $[0,1]$ with the sup norm and let $C[0,1]$ also have the sup norm. Define $T\colon X \to C[0,1]$ by $(Tf)(t) = f'(t)$. Then $T$ is linear but is not bounded: find a sequence $(f_n)$ of continuously differentiable functions on $[0,1]$ for which $\|f_n\|_\infty \leqslant 1$ for all $n$ but $\|f_n'\|_\infty \to \infty$.

[*Here $X$ is not a Banach space: it is a dense proper subspace of $C[0,1]$. Results on Banach spaces proved in FA-II show why unbounded operators on Banach spaces are hard to find.*]

3.5. **Remarks on calculating operator norms.** Suppose $X, Y$ are normed spaces and $T\colon X \to Y$ is linear. Suppose we seek to show $T$ is bounded and to calculate its norm.

As our examples have shown, it is often relatively easy to find *some* constant $M$ such that $\|Tx\| \leqslant M\|x\|$ for all $x$, This implies that $T$ is bounded, with $\|T\| \leqslant M$. But it is often harder to find the *least* such $M$, as in (Op4) in Proposition 3.1.

Two cases can arise.

1. **Supremum attained in (Op1)/(Op2)/(Op3).** We are in luck! We can show

   $\|Tx\| \leqslant M\|x\|$ for all $x$ and that there exists $x_0 \neq 0$ such that $\|Tx_0\| = M\|x_0\|$.

   Then the sup in (Op1) is attained and we have $\|T\| = M$.
2. **Supremum not (necessarily) attained in (Op1)/(Op2/Op3).**
   Recall the **Approximation Property** for sups from Prelims Analysis I: *Let $S$ be a non-empty subset of $\mathbb{R}$ which is bounded above, so* $\sup S$ *exists. Then, given $\varepsilon > 0$, there exists $s \in S$, depending on $\varepsilon$, such that*

   $$\sup S - \varepsilon < s \leqslant \sup S.$$

   This characterises the sup in the sense that if $M$ is an upper bound for $S$ and there exists a sequence $(s_n)$ in $S$ such that $s_n \geqslant M - n^{-1}$ for all $n$ then $M = \sup S$. Our witnessing sequence method in 3.3(3) draws on this. Note also Example 3.6.

Note finally that we can also use sequences to witness that a linear map $T$ is unbounded: this happens if there exists a sequence $(x_n)$ with $\|x_n\| = 1$ (or $(\|x_n\|)$ bounded will do) and $\|Tx_n\| \to \infty$ as $n \to \infty$.

3.6. **Example: bounded linear operators on sequence spaces.**

By way of illustration, consider a given bounded linear operator $T\colon \ell^1 \to Y$, where $Y$ is some normed space. We can get information about $\|T\|$ by considering the images $Te_k$, where $e_k = (\delta_{kj})_{j\geqslant 1}$, for $k \geqslant 1$. For any $k$, we have $\|e_k\|_1 = 1$ and hence

$$\sup_k \|Te_k\| \leqslant \|T\|.$$

We claim there is equality. Take any $x = (x_j)$ in $\ell^1$ and define

$$x^{(n)} = \sum_{k=1}^{n} x_k e_k = (x_1, \ldots, x_n, 0, 0, \ldots).$$

Then

$$\|x - x^{(n)}\|_1 = \sum_{k=n+1}^{\infty} |x_k| \to 0 \text{ as } n \to \infty.$$

Hence, because $T$ is continuous, $Tx^{(n)} \to Tx$ in $Y$. But

$$Tx^{(n)} = T\left(\sum_{k=1}^n x_k e_k\right) = \sum_{k=1}^n x_k T e_k.$$

It follows that, for all $n$,

$$\begin{aligned}\|Tx^{(n)}\| &\leqslant \textstyle\sum_{k=1}^n |x_k| \|Te_k\| \leqslant \sup_{1\leqslant k\leqslant n} \|Te_k\| \sum_{k=1}^n |x_k| \\ &\leqslant (\sup_k \|Te_k\|) \textstyle\sum_{k=1}^\infty |x_k| = (\sup_k \|Te_k\|)\, \|x\|_i.\end{aligned}$$

By continuity of the norm in $Y$,

$$\|Tx\| = \lim_{n\to\infty} \|Tx^{(n)}\| \leqslant (\sup \|Te_k\|)\|x\|_1.$$

This proves our claim that

$$\|T\| = \sup_{k\geqslant 1} \|Te_k\|.$$

Specialising to $Y = \ell^p$, for $1 \leqslant p < \infty$, we deduce that, for any $T \in \mathcal{B}(\ell^1, \ell^p)$,

$$\|T\| = \sup_{k\geqslant 1} \|Te_k\|_p.$$

See Problem sheet Q, 11 for a cautionary example concerning a bounded linear operator from $\ell^\infty$ into $\ell^1$.

3.7. **Aside: matrix norms** (attn: numerical analysts)**.**

Norms on vector spaces $M_n(\mathbb{R})$ are useful in numerical linear algebra, for example in error analysis of solutions of linear equations. To be specific, let us consider $X = \mathbb{R}^n$ with the 1-norm, 2-norm or $\infty$-norm.

Take the standard basis $\{e_1, \dots, e_n\}$ for $\mathbb{R}^n$. Then any linear map $T\colon X \to X$ is represented with respect to this basis by a matrix $A = (a_{ij})$ with

$$Te_k = \sum_{k=1}^n a_{ik} e_i.$$

Then one may measure the 'size' of $A$ using either of the following quantities:

$$\|A\|_1 = \max_{1\leqslant j\leqslant n} \left( \sum_{i=1}^n |a_{ij}| \right),$$

$$\|A\|_\infty = \max_{1\leqslant i\leqslant n} \left( \sum_{j=1}^n |a_{ij}| \right).$$

It can be shown, cf. Example 3.6, that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are respectively the norms on $\mathcal{B}(\mathbb{R}^n)$ when $\mathbb{R}^n$ has the 1-norm and the $\infty$-norm.

One may likewise ask for a formula for the norm on $\mathcal{B}(\mathbb{R}^n)$ when $\mathbb{R}^n$ has the 2-norm. This is more elusive. However when we have a real symmetric matrix $A$, this is diagonalisable, with eigenvalues $\lambda_1, \dots, \lambda_n$ (not necessarily distinct). Then

$$\|A\|_2 := \sqrt{\sup_{\|x\|_2=1} \|Ax\|_2^2} = \max\{|\lambda_1|. \dots, |\lambda_n|\}.$$

We now return to general theory.

3.8. **Theorem (completeness).** *Let $X$ be a normed space and $Y$ a Banach space, both over the same field. Then the normed space $\mathcal{B}(X,Y)$ is complete.*

*Proof.* The proof goes in very much the same way as the completeness proofs for function spaces given in Section 2. Let $(T_n)$ be a Cauchy sequence in $\mathcal{B}(X,Y)$.

**Step 1: candidate limit.** For each $x$ we have

$$\|T_m x - T_n x\| = \|(T_m - T_n)x\| \leqslant \|T_m - T_n\|\,\|x\|.$$

We deduce that $(T_n x)$ is Cauchy in $Y$, and so convergent to some $Tx \in Y$. We thus have a map $T\colon X \to Y$. It follows from the continuity of addition and scalar multiplication (see 2.2) that $T$ is linear.

**Steps 2 and 3: from pointwise convergence to norm convergence and proof that $T \in \mathcal{B}(X,Y)$.**

Observe that if, for some $N$, we can show that the linear map $T - T_N$ is bounded then $T = (T - T_N) + T_N$ will be bounded too. For any fixed $\varepsilon > 0$,

$$\exists N\ \forall m,n \geqslant N \quad \|T_m x - T_n x\| \leqslant \|T_m - T_n\|\|x\| \leqslant \varepsilon\|x\|.$$

Fix $x$ and $n \geqslant N$ and let $m \to \infty$ to get $\|Tx - T_n x\| \leqslant \varepsilon\|x\|$ and hence $\|(T - T_n)x\| \leqslant \varepsilon\|x\|$. This implies in particular that $T - T_N$ is bounded. Also $n \geqslant N$ implies $\|T - T_n\| \leqslant \varepsilon$. Therefore $\|T - T_n\| \to 0$. □

3.9. **Corollary.** *For any normed space $X$ over $\mathbf{F}$, with $\mathbb{F}$ as $\mathbb{R}$ or $\mathbb{C}$, the space $X^* = \mathcal{B}(X,\mathbb{F})$ of bounded linear functionals is complete, where the norm is*

$$\|f\| = \sup\left\{ \frac{|f(x)|}{\|x\|} \mid x \neq 0 \right\}.$$

We shall make significant use of spaces of the form $X^*$ later in the course.

3.10. **Products of linear operators.**

We can form compositions of bounded linear operators in just the same way as we can form composite linear maps between vector spaces. Let $T \in \mathcal{B}(X,Y)$ and $S \in \mathcal{B}(Y,Z)$, where $X, Y, Z$ are normed spaces over $\mathbb{F}$. Then $ST$, defined by $(ST)(x) = S(Tx)$ for all $x \in X$, is a member of $\mathcal{B}(X,Z)$. Boundedness of $ST$ comes from

$$\|(ST)x\| = \|S(Tx)\| \leqslant \|S\|\,\|Tx\| \leqslant \|S\|\,\|T\|\,\|x\|.$$

Moreover we deduce from this that

$$\|ST\| \leqslant \|S\|\,\|T\|.$$

Assume $T \in \mathcal{B}(X)$. Then, by induction, $T^n \in \mathcal{B}(X)$ and $\|T^n\| \leqslant \|T\|^n$, for $n = 1, 2, \ldots$ .

3.11. **Kernel and image.**

Let $X$ and $Y$ be normed spaces. Let $T \in \mathcal{B}(X,Y)$, that is, $T$ is linear and is continuous, *alias* bounded.

In Linear Algebra in finite-dimensional spaces, the kernel and image of a linear map feature prominently. They are still important, but less central, when we work in infinite-dimensional normed spaces. But it is worth noting that the subspace

$$\ker T := \{\, x \in X \mid Tx = 0 \,\}$$

equals $T^{-1}\{0\}$. This is the inverse image under a continuous map of a closed set and so is closed.

On the other hand, $TX = \operatorname{im} T$ is not in general closed. See Example 3.16 for an instance of this.

3.12. **Examples: kernel and image.**

Recall the shift operators $R$ and $L$ in Example 3.3(1):

$$R(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots),$$
$$L(x_1, x_2, \ldots) = (x_2, x_3, \ldots).$$

These belong to $\mathcal{B}(\ell^1)$.

It is easy to see that $R$ is injective but not surjective and that $L$ is surjective but not injective: its kernel is the space spanned by $e_1 = (1, 0, 0, \ldots)$. Of course, this is behaviour that cannot arise for a linear map from a finite-dimensional space to itself, thanks to the Rank-Nullity Theorem.

3.13. **Invertibility of a bounded linear operator.**

Suppose we are given a bounded linear operator $T \in \mathcal{B}(X)$ where $X$ is a normed space. We would like to know when $T$ has an inverse $S$ *which is again a bounded linear operator*. So we seek $S \in \mathcal{B}(X)$ such that $S \circ T = T \circ S = I$. For this it is necessary that $T$ be a bijection, but in general this will not be sufficient. Note too that we demand a two-sided inverse.

When such an $S$ exists we write $S = T^{-1}$.

3.14. **A checklist for invertibility of $T \in \mathcal{B}(X)$ ($X$ a normed space).**

Mindful of Example 8.3 we now pick apart what we need in order that a bounded linear operator is invertible: We must have

(a) $T$ injective (equivalently $\ker T = \{0\}$);

(b) $T$ surjective, that is, $TX = X$.

Assume now that (a) and (b) hold. Then there exists a *map* $S \colon X \to X$ such that $T \circ S = S \circ T = I$. We'd then want $S$ to be a bounded linear operator on $X$. So we need

(c) $S$ is linear: this is always true (routine calculation, just Linear Algebra);

(d) $S$ is bounded, that is, there exists a constant $K > 0$ such that

$$\|Sy\| \leqslant K\|y\| \quad \forall y \in X.$$

Given (b), this is equivalent to

$$(\star) \qquad \exists \delta > 0 \text{ such that } \|Tx\| \geqslant \delta\|x\| \quad \forall x \in X.$$

When this holds $K := \delta^{-1}$ provides a bound for $\|T^{-1}\|$.

In practice, the kernel of $T$ can usually be found by direct calculation. Surjectivity of $T$ is generally much harder to check. A fallback strategy is to aim to show that $TX$ is both closed and dense in $X$, from which $TX = X$ would follow.

The following proposition is informative and will be useful later. Note that we need to assume that $X$ is a Banach space.

3.15. **Proposition (closed range).** *Let $X$ be a Banach space and $T \in \mathcal{B}(X)$. Assume that $(\star)$ holds for some constant $K$. Then $T$ is injective and $TX$ is closed.*

*If in addition $\overline{TX} = X$, then $T$ is bijective and $T$ is invertible.*

*Proof.* From $(\star)$, $Tx = 0$ implies $x = 0$, so $\ker T = \{0\}$. Now assume $(y_n)$ is a sequence such that $y_n = Tx_n$ and $y_n \to y$. Then $(Tx_n)$ is Cauchy and then $(\star)$ implies that $(x_n)$ is Cauchy. Since $X$ is Banach, there exists $x$ such that $x_n \to x$. Then $y_n = Tx_n \to Tx$ so $y \in TX$.

Assume further that $TX$ is dense. Then $TX = X$. Moreover $(\star)$ tells us that $S = T^{-1}$ is bounded. □

3.16. **Examples: invertible and non-invertible operators.**

(1) Let $K$ be a compact subset of $\mathbb{C}$ and consider $X = C(K)$, the space of complex-valued continuous functions on $K$ with the sup norm. Define $T \in \mathcal{B}(X)$ by

$$(Tf)(z) = zf(z) \qquad (z \in K,\ f \in X).$$

Then $T$ is invertible if and only if $0 \notin K$. For the $\Longleftarrow$ direction, observe that $0 \notin K$ implies that $T$ is a bijection with inverse $(T^{-1}g)(z) = g(z)/z$ and that $\|T^{-1}g\| \leqslant \|z\|/\delta$ where $\delta := \operatorname{dist}(0, K) > 0$, so $T^{-1}$ is bounded.

(2) Let $T \in \mathcal{B}(c_0)$ be given by $T(x_j) = (x_j/j)$. Then $\ker T = \{0\}$.

The operator $T$ is not surjective: no $(x_j) \in c_0$ exists such that $T(x_j) = (1/\sqrt{j})$. However the range $Tc_0$ is dense since it contains $c_{00}$ the subspace of all sequences with only finitely many non-zero coordinates.

Also $\|e_n\| = 1$ and $\|Te_n\| = 1/n$. It follows that $(\star)$ in 3.14 cannot hold.

Our final invertibility result in this section gives a sufficient condition for an operator on a Banach space to be invertible and which directly provides a formula for the inverse. Note how completeness is used in the proof.

3.17. **Proposition.** *Let $X$ be a Banach space. Let $T \in \mathcal{B}(X)$ be such that $\|T\| < 1$. Then $I - T$ is invertible with inverse given by $\sum_{k=0}^{\infty} T^k$ (where $T^0 := I$).*

*Moreover, if $P \in \mathcal{B}(X)$ is such that $\|I - P\| < 1$ then $P$ is invertible.*

*Proof.* Assume $\|T\| < 1$. Define

$$S_n = \left(I + T + T^2 + \cdots + T^n\right).$$

Since $\|T^k\| \leqslant \|T\|^k$ for all $k$, the series $\sum \|T^k\|$ converges. Note $\mathcal{B}(X) := \mathcal{B}(X, X)$ is a Banach space because $X$ is, by 3.8. Now by Theorem 2.10 there exists $S \in \mathcal{B}(X)$ such that $\|S_n - S\| \to 0$. Also

$$(I - T)S_n = S_n(I - T) = I - T^{n+1}.$$

Letting $n \to \infty$, noting that $\|T^{n+1}\| \to 0$, we see that $(I - T)^{-1}$ exists and equals $S$.

For the final assertion put $T = I - P$ and note that $P = I - T$. □

This result is important in spectral theory and in its applications, for example the theory of integral equations.

## 4. Finite-dimensional normed spaces

*In which we reveal why linear algebra in finite-dimensional real or complex vector spaces doesn't normally explicitly involve analysis, and we show how finite-dimensional spaces fit into functional analysis.*

*The key results are*

- *any two norms on a given finite-dimensional normed space are equivalent;*
- *any linear operator with finite-dimensional domain is bounded;*
- *any finite-dimensional normed space is a Banach space;*
- *any finite-dimensional subspace of a normed space is closed.*

### 4.1. Finite-dimensional spaces, algebraically.

Let $X$ be a vector space over $\mathbb{F}$, where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$. Assume that $X$ is finite-dimensional, of dimension $m$, and fix some basis $\{x_1, \ldots, x_m\}$. From Prelims Linear Algebra, the map

$$P\colon (\lambda_1 x_1 + \cdots + \lambda_m x_m) \mapsto (\lambda_1, \ldots, \lambda_m)$$

is well-defined and sets up an isomorphism (that is, a linear bijection) from $X$ onto $\mathbb{F}^m$. Denote its inverse by $Q$. Note (Prelims LA result) that $Q$ is necessarily linear.

4.2. **Theorem (introducing topology).** *Let $X$ be a vector space over $\mathbb{F}$ of dimension $m$, equipped with a norm $\|\cdot\|$. Let $Q\colon \mathbb{F}^m \to X$ be the isomorphism set up in 4.1 and give $\mathbb{F}^m$ the $\infty$-norm. Then*

(i) *$Q$ is bounded and so continuous;*
(ii) *$Q$ is a homeomorphism and hence so is $P$.*

*Proof.* (i) Case $m = 1$: Here $Q\colon \lambda_1 \mapsto \lambda_1 x_1$ with $x_1$ necessarily non-zero. Trivially $Q$ is a homeomorphism.

Case $m > 1$:

$$Q\colon (\lambda_1, \ldots, \lambda_m) \mapsto \lambda_1 x_1 + \ldots + \lambda_m x_m.$$

For each $j \in \{1, \ldots, m\}$ there is a projection map $\pi_j\colon (\lambda_1, \ldots, \lambda_m) \mapsto \lambda_j$ which is a bounded linear operator from $\mathbb{F}^m$ to $\mathbb{F}$ (it has norm $\leqslant 1$). Now consider for each $j$ the composite map $Q_j \circ \pi_j$, where $Q_j\colon \lambda_j \mapsto \lambda_j x_j$:

$$(\lambda_1, \ldots, \lambda_m) \overset{\pi_j}{\longmapsto} \lambda_j \overset{Q_j}{\longmapsto} \lambda_j x_j.$$

As noted for the case $m = 1$, each $Q_j$ is continuous. Hence $Q_j \circ \pi_j\colon \mathbb{F}^m \to X$ is continuous. Finally note that $Q = (Q_1 \circ \pi_1) + \cdots + (Q_m \circ \pi_m)\colon \mathbb{F}^m \to X$ is continuous too.

(ii) Consider $C := \{\, y \in \mathbb{F}^m \mid \|y\|_\infty = 1 \,\}$. The set $C$ is closed and bounded, and hence compact by the Heine–Borel Theorem. Therefore $Q(C)$ is compact since $Q$ is continuous. We want to show the linear bijection $P = Q^{-1}\colon X \to \mathbb{F}^m$ is continuous. Certainly $0 \notin Q(C)$ since $0 \notin C$ and $Q$ is a linear bijection. As a compact subset of a metric space $Q(C)$ is necessarily closed in $X$. Hence there exists $\delta > 0$ such that $\|x\| < \delta$ implies $x \notin Q(C)$. Rephrasing this,

$$\|y\|_\infty = 1 \Longrightarrow Qy \notin \{\, x \in X \mid \|x\| < \delta \,\} \implies \|Qy\| \geqslant \delta.$$

Since $Q$ is linear and surjective this gives $\|Px\| \leqslant \delta^{-1}\|x\|$ for all $x \in X$, so $P$ is continuous, as required. (Recall $(\star)$ in 3.14.) $\square$

We already know that the various $p$-norms on $\mathbb{F}^m$ are equivalent. But we can now do better.

4.3. **Theorem (equivalence of norms).** *Any two norms on a finite-dimensional normed space $X$ are equivalent.*

*Proof.* Let $\|\cdot\|$ and $\|\cdot\|'$ be norms on $X$. Then there exist homeomorphisms

$$P\colon (X, \|\cdot\|) \to (\mathbb{F}^m, \|\cdot\|_\infty) \quad \text{and} \quad P'\colon (X, \|\cdot\|') \to (\mathbb{F}^m, \|\cdot\|_\infty),$$

by Theorem 4.2, and these are equal as set maps. Hence the identity maps

$$(P')^{-1} \circ P\colon (X, \|\cdot\|) \to (X, \|\cdot\|') \quad \text{and} \quad P^{-1} \circ P'\colon (X, \|\cdot\|') \to (X, \|\cdot\|)$$

are both continuous and are obviously linear. As such, they are bounded linear operators, and this is the same as saying that the norms $\|\cdot\|$ and $\|\cdot\|'$ are equivalent. □

4.4. **Corollary (boundedness of linear operators on a finite-dimensional normed space).** *Let $X$ and $Y$ be normed spaces over $\mathbb{F}$, with $X$ finite-dimensional. Assume that $T\colon X \to Y$ is linear. Then $T$ is bounded.*

*Assume $X = Y$. A linear map $T\colon X \to X$ is invertible if and only if it is a bijection. Here invertibility is defined as in 3.13.*

*Proof.* Denote the given norm of $X$ by $\|\cdot\|$. Take a fixed basis $\{x_1, \ldots, x_m\}$ for $X$ and let $x = \sum \lambda_j x_j$ be a general element of $X$, where the scalars $\lambda_j$ are uniquely determined by $x$. Then

$$\begin{aligned}\|T(\lambda_1 x_1 + \cdots + \lambda_m x_m)\| &= \|\lambda_1 T x_1 + \cdots + \lambda_m T x_m\| \\ &\leqslant |\lambda_1|\|Tx_1\| + \cdots + |\lambda_m|\|Tx_m\| \leqslant \Big(\sum_{j=1}^m \|Tx_j\|\Big) \max_{1\leqslant j\leqslant m} |\lambda_j|.\end{aligned}$$

Define a new norm $\|\cdot\|'$ on $X$ by

$$\|x\|' = \|(\lambda_1 x_1 + \ldots + \lambda_m x_m)\|' := \max_{1\leqslant j\leqslant m} |\lambda_j|.$$

By Theorem 4.3, $\|\cdot\|$ and $\|\cdot\|'$ are equivalent. This implies that there is a constant $K$ such that $\|x\|' \leqslant K\|x\|$ for all $x$. Therefore

$$\|Tx\| \leqslant K\Big(\sum_{j=1}^m \|Tx_j\|\Big)\|x\| \quad (\text{for all } x \in X).$$

We conclude that $T$ is bounded. □

4.5. **Theorem (completeness of finite-dimensional normed spaces).**

(i) $(\mathbb{F}^m, \|\cdot\|_\infty)$ *is a Banach space.*

(ii) *Let $(X, \|\cdot\|)$ be a finite-dimensional normed space. Then $X$ is a Banach space.*

*Proof.* (i) This is a very simple instance of the strategy for proving completeness in function spaces (we can regard $\mathbb{F}^m$ as the space of bounded functions from $\{1, \ldots, m\}$ to $\mathbb{F}$, or just think in terms of coordinates).

For (ii) it suffices to prove that $X$ is complete in *some* norm since all norms on $X$ are equivalent and, for equivalent norms, the same sequences are Cauchy. Take a basis $\{x_1, \ldots, x_m\}$ for $X$ and *define* $\|\sum_j \lambda_j x_j\|$ to be $\max_j |\lambda_j|$. This is a norm on $X$ and $X$ with this norm is Banach, exactly as in (i). □

We now consider closed subspaces, recalling Proposition 0.12. We give two proofs for the following result, one which uses all the machinery of this section, the other much more direct.

4.6. **Proposition.** *Let $Y$ be a finite-dimensional subspace of a normed space $X$. Then $Y$ is closed.*

*Proof.* **Method 1**: $(Y, \|\cdot\|_X)$ is a finite-dimensional normed space, so complete by Theorem 4.5(ii). Hence $Y$ is closed by Proposition 0.12.

**Method 2**: We proceed by induction on $\dim Y$. If $\dim Y = 0$ then $Y = \{0\}$ and this is closed.

Now assume that $Z$ is any closed subspace of $X$ and let $y \in X \setminus Z$. We claim that $W = \operatorname{span}(Z \cup \{y\})$ is closed. Each element of $W$ is expressible (uniquely) as $z + \lambda y$, where $z \in Z$ and $\lambda \in \mathbb{F}$. Suppose $w_n \to x$ where $w_n = z_n + \lambda_n y$.

**Case 1:** $(\lambda_n)$ **is unbounded.** By passing to a subsequence if necessary we may assume that $|\lambda_n| \to \infty$ (and wlog $\lambda_n \neq 0$). Then

$$|\lambda_n|^{-1}\|w_n\| = \|\lambda_n^{-1} z_n + y\|.$$

Letting $n \to \infty$, noting that $(w_n)$ is norm-bounded, we see that $y \in \overline{Z} = Z$, which is a contradiction.

**Case 2:** $(\lambda_n)$ **is bounded.** By Prelims Analysis, $(\lambda_n)$ has a convergent subsequence, $(\lambda_{n_r})$ say, converging to some $\lambda \in \mathbb{F}$. Now

$$z_{n_r} = w_{n_r} - \lambda_{n_r} y \to w - \lambda y.$$

But this implies that $(z_{n_r})$ converges to some $z$, necessarily in $Z$. Finally we get $w \in W$, so $W$ is closed. $\square$

The famous Heine–Borel Theorem tells us that in a finite-dimensional normed space $\mathbb{F}^m$ compactness is equivalent to (closed + bounded). The following result tells us that the backward implication fails in any *infinite-dimensional* normed space.

4.7. **Theorem (compactness of closed unit ball).** *Let $X$ be a normed space and let $S := \overline{\mathrm{B}}(0,1) = \{\, x \in X \mid \|x\| \leqslant 1 \,\}$ be the closed unit ball in $X$. Then $S$ is compact if and only if $X$ is finite-dimensional.*

*Proof.* $\Longleftarrow$: By Theorem 4.2, $X$ is isomorphic and homeomorphic to some normed space $\mathbb{F}^m$. Thus it is sufficient to show that the closed unit ball in a space $\mathbb{F}^m$ is compact. But as noted above in this special case this follows from the Heine–Borel Theorem.

$\Longrightarrow$: We first introduce some useful notation, which just uses the vector space structure of $X$. For $A, B \subseteq X$ and $c > 0$, let

$$A + B = \{\, a + b \mid a \in A,\, b \in B \,\} \quad \text{and} \quad cA = \{\, ca \mid a \in A \,\}.$$

Note that if $Y$ is a subspace then $Y + Y = Y$ and $cY = Y$ for all $c > 0$.

Now assume that $S$ is compact. We shall make use of the fact that a compact subset of a metric space is **totally bounded**. This means that for each $\varepsilon > 0$ there exists a *finite* set $F_\varepsilon$ (called an $\varepsilon$-net) in $X$ such that every point of $S$ is within a distance $\varepsilon$ of some point in $F_\varepsilon$. The proof is easy: consider the open cover of $S$ consisting of all open balls in $X$ of radius $< \varepsilon$ with centres in $S$.

Let $F$ be a $1/2$-net. Then, for any $x \in S$, there exists $u \in F$ such that $\|x - u\| < 1/2$. Saying this another way, $S \subseteq F + \mathrm{B}(0, 1/2)$, and this implies $S \subseteq F + \frac{1}{2}S$.

Let $Y = \operatorname{span}(F)$. Then

$$S \subseteq F + \tfrac{1}{2}S \subseteq Y + \tfrac{1}{2}S.$$

By a simple notation-chase,
$$\tfrac{1}{2}S \subseteq Y + \tfrac{1}{4}S.$$
Putting these together
$$S \subseteq Y + Y + \tfrac{1}{4}S = Y + \tfrac{1}{4}S.$$
Proceeding by induction
$$S \subseteq Y + 2^{-k}S \text{ for all } k \geqslant 1.$$
But then $S \subseteq \bigcap_k (Y + 2^{-k}S)$. The set on the RHS is $\overline{Y}$. Since $Y$ is finite-dimensional, $Y$ is closed. Therefore $S \subseteq Y$. But this implies $X \subseteq Y$. Hence $X = Y$ and so $X$ is finite-dimensional. □

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## 5. Density and separability

*In which we reveal different aspects of the notion of density. Major results:*

- *Stone–Weierstrass Theorem 5.10, on dense subspaces of $(C(K), \|\cdot\|_\infty)$, where $K$ is compact;*
- *Theorem 5.25, which is a technically useful general theorem on extending a bounded linear operator from a dense subspace to the whole space.*

*In addition we discuss the notion of separability. A separable space is one which has a countable dense subset. Many of our familiar examples are shown to be separable, a few are inseparable.*

### 5.1. **Spaces of continuous functions on compact sets.**

Our goal is to identify amenable dense subsets in spaces of continuous functions, for the sup norm. In particular we'll be interested in polynomial approximations to continuous functions on closed bounded subintervals of $\mathbb{R}$.

For definiteness, let's consider first $C[0,1]$, the continuous real-valued functions on $[0,1]$. Then $C[0,1]$ is a normed space for the sup norm, with convergence being uniform convergence (and $C[0,1]$ is in fact a Banach space, though this is not directly relevant in this section).

We know that $C[0,1]$ supports pointwise-defined operations as follows:

(1) addition and scalar multiplication;
(2) a commutative multiplication, with $\|fg\| \leqslant \|f\|\,\|g\|$, and with a unit for multiplication, *viz.* the constant function $\mathbf{1}$;
(3) modulus, $|f|$, for $f \in C[0,1]$;
(4) max and min: for $f, g \in C[0,1]$,
$$f \vee g := \max\{f, g\}; \quad f \wedge g := \min\{f, g\}.$$

Just as the algebraic operations and modulus rely for their definitions on the existence of corresponding operations in $\mathbb{R}$, so too the order-theoretic operations in (4) rely on the corresponding order-theoretic operations in $\mathbb{R}$ and so on the underlying order relation $\leqslant$ on $\mathbb{R}$. A consequence is that, while (1)–(3) would go through unchanged to complex-valued functions, (4) would not.

Selecting particular features:

- $C[0,1]$ is a *commutative ring with identity* ((1), with scalar multiplication ignored, & (2));
- $C[0,1]$ is a *commutative Banach algebra* ((1) & (2), plus completeness to cover the inclusion of 'Banach' in the name).
- $C[0,1]$ is a *vector lattice*: ((1) & (4)).

A surfeit of riches!

[**Note**: In all cases the formal definitions of the italicised terms involve some axioms to ensure the various operations interact as we would wish. Not important for us because we only work with the special case of $C(K)$.]

Observe that we have also met normed spaces with a supplementary operation of product in Section 3: the spaces $\mathcal{B}(X,Y)$, in which $\|ST\| \leqslant \|S\|\,\|T\|$ and which are Banach spaces when $Y$ is a Banach space. These spaces are examples of *non-commutative Banach algebras*.

Everything said so far works without change if we replace $[0,1]$ by any compact space $K$. Compactness of $K$ guarantees that $\|f\|_\infty$ is finite for each $f \in C(K)$ so the sup norm is available. Henceforth in this section $C(K)$ will denote the real-valued continuous functions on a non-empty compact set $K$.

### 5.2. **Separation of points.**

We say that a set $Y \subseteq C(K)$ **separates the points of** $K$ if, given $p, q \in K$ with $p \neq q$ there exists $g \in Y$ such that $g(p) \neq g(q)$. As first examples we note that, in $C[0,1]$, each of the following sets separates the points of $[0,1]$:

$C[0,1]$, the polynomials, the piecewise-linear functions.

Obviously, the constant functions fail to separate points. In general, the smaller a subset $Y$ of $C(K)$ is the less likely it is to separate points. We may ask when $C(K)$ itself separates the points of $K$. This is the case if

(i) $K = [0,1]$ or more generally any closed bounded interval in $\mathbb{R}$;
(ii) $K$ is a compact subset in $\mathbb{R}^n$ (exercise);
(iii) $K$ is a compact subset of a metric space (use the fact that for each fixed point $a$, the map $x \mapsto d(x,a)$ is continuous);
(iv) $K$ is a compact Hausdorff space (quite advanced topology, based on Urysohn's Lemma; mentioned here just to complete the picture).

We mention these results because the Stone–Weierstrass Theorem that we prove for a space $C(K)$ (both forms) would be vacuous if $C(K)$ failed to separate points.

What we are aiming for is sufficient conditions on a subspace $Y$ of $C(K)$ which will guarantee that it is dense in $C(K)$. This suggests that it needs to be 'big'. In particular it's worth noting that $Y$ cannot be dense if it is characterised by a property which lifts from $Y$ to its closure but which does not hold universally in $C(K)$. For example, the following are proper closed subspaces of $C[0,1]$ and so cannot be dense:

- $\{ f \in C[0,1] \mid f(1/2) = 0 \}$;
- $\{ f \in C[0,1] \mid f(0) = f(1) \}$.

The first of these separates points but fails to contain the constants, while the second contains the constants but fails to separate points.

5.3. **Two-point fit lemma.** *Let $Y$ be a subspace of $C(K)$ containing the constant functions and separating the points of $K$. Let $p \neq q$ in $K$ and $\alpha, \beta \in \mathbb{R}$. Then there exists $g \in Y$ such that*

$$g(p) = \alpha \quad \text{and} \quad g(q) = \beta.$$

*Proof.* Since $Y$ separates points, we can first choose $f \in Y$ such that $f(p) \neq f(q)$. Now consider $g = \lambda f + \mu \mathbf{1}$ and aim to choose $\lambda, \mu \in \mathbb{R}$ so that

$$\alpha = \lambda f(p) + \mu, \qquad \beta = \lambda f(q) + \mu.$$

These equations are uniquely soluble for $\lambda$ and $\mu$. Since $Y$ is a subspace containing the constants, $\lambda f + \mu \mathbf{1} \in Y$. □

5.4. **More about lattice operations in $C(K)$.**

A subspace $L$ of $C(K)$ is said to be a **linear sublattice** if $f, g \in L$ implies $f \vee g$ and $f \wedge g$ belong to $L$. It is easy to see that $L$ is a linear sublattice if and only if $f \in L$ implies $|f| \in L$. This follows from the (familiar) formulae:

$$|f| = (f \vee 0) + ((-f) \vee 0);$$
$$f \vee g = \tfrac{1}{2}\left(f + g + |f - g|\right), \qquad f \wedge g = \tfrac{1}{2}\left(f + g - |f - g|\right).$$

Note that it is crucial here that $L$ be a subspace.

Note too that we can form the max and the min of any *finite* number of continuous real-valued functions and that these are again continuous. But this doesn't extend to infinite families of functions.

Now take a linear sublattice $L$ of $C(K)$ which contains the constants and separates points. Then the Two-point fit Lemma implies that, for a given $f$ in $C(K)$, there is a member of $L$ which equals $f$ at any two specified points of $K$. This, thanks to continuity, should enable us to approximate $f$ locally by elements of $L$. To get a global approximation to $f$, we first seek to use max operations to approximate $f$ from below and then to use min operations to approximate from above. We will need to call on compactness of $K$ to ensure that we only need a *finite* number of approximating functions from $L$ at each of these two stages. [*Those who have seen the proof that a compact Hausdorff space is normal may recognise similarities.*]

So here's our first density theorem, whose proof follows the strategy outlined above.

5.5. **Stone–Weierstrass Theorem (real case, lattice form).** *Let $L$ be a subspace of $C(K)$ which is such that*

(i) *$L$ is a linear sublattice;*
(ii) *$L$ contains the constant functions;*
(iii) *$L$ separates the points of $K$.*

*Then $L$ is dense in $C(K)$.*

*Proof.* We want to show that, given $f \in C(K)$ and any $\varepsilon > 0$ we can find $h \in C(K)$ such that

$$f - \varepsilon < h < f + \varepsilon.$$

**Step 1: approximating $f$ at points $p, q \in K$.** We claim there exists $g \in L$ such that $g(p) = f(p)$ and $g(q) = f(q)$. If $p \neq q$ this comes from the Two-point fit Lemma. If $p = q$ a constant function with serve for $g$.

[*This step doesn't need the assumption that $L$ is closed under max and min.*]

**Step 2: approximating $f$ from below.** Fix $p \in K$ and let $q \in K$. Use Step 1 to construct $g_q \in L$ such that $f(p) = g_q(p)$ and $f(q) = g_q(q)$. By continuity of $g_q - f$ there exists an open set $U_q \ni q$ such that

$$g_q(s) > f(s) - \varepsilon \quad \text{for all } s \in U_q.$$

By compactness of $K$ there exist $q_1, \ldots, q_n \in K$ such that $K = U_{q_1} \cup \cdots \cup U_{q_n}$. Then

$$g := g_{q_1} \vee \cdots \vee g_{q_n} > f - \varepsilon$$

and $g(p) = f(p)$. Since $g$ depends on $p$, let's now denote it $g^p$.

**Step 3: approximating $f$ from above.** We now vary $p$. For each $p$ we can choose an open set $V_p$ containing $p$ such that

$$g^p(t0 < f(t) + \varepsilon \quad \text{for all } t \in V_p.$$

By compactness, there exists $p_1, \ldots, p_m$ such that $K = V_{p_1} \cup \cdots \cup V_{p_m}$. Now

$$h := g^{p_1} \wedge \cdots \wedge g^{p_m} < f + \varepsilon.$$

**Step 4: putting the pieces together.** Consider $h$ as in Step 3. From Step 2, each $g^{p_i} > f - \varepsilon$. Hence $h > f - \varepsilon$. Therefore $\|h - f\|_\infty < \varepsilon$. Also $h \in L$. We conclude that $f \in \overline{L}$. □

5.6. **Example: an application of SWT, lattice form.**

The set $L$ of piecewise-linear real-valued continuous functions on a closed bounded interval $[a, b] \in \mathbb{R}$ form a linear sublattice of $C[a, b]$ (why?) and $L$ contains the constants and separates points. By 5.5, $L$ is dense in $C[a, b]$. (This can also be proved directly, by an $\varepsilon - \delta$ proof involving uniform continuity.)

We shall now use the sublattice form of SWT to get at a different form of the theorem which is particularly useful, and which subsumes Weierstrass's classic polynomial approximation theorem. We are going to need the following lemma, which can be seen as a very special case of the uniform approximation of a continuous function on $[0, 1]$ by polynomials.

5.7. **Technical lemma (approximating $\sqrt{t}$ on $[0, 1]$ by polynomials).** *Define a sequence $(p_n)$ of polynomials recursively by*

$$p_1(t) = 0, \quad p_{n+1}(t) = p_n(t) + \frac{1}{2}(t - p_n(t)^2) \quad (n \geqslant 1).$$

*Then $(p_n(t))$ is an increasing sequence for each $t \in [0, 1]$ and $p_n(t) \xrightarrow{u} t^{1/2}$ on $[0, 1]$.*

*Proof.* We can prove easily that if

$$0 = p_1(t) \leqslant p_2(t) \leqslant \cdots \leqslant p_k(t) \leqslant t^{1/2} \quad (\text{for all } t \in [0, 1])$$

holds for $k = n$ then it holds for $k = n + 1$ too, and it is certainly true for $k = 1$. Hence by induction $(p_n(t))$ is a monotonic increasing sequence bounded above by 1. Therefore $(p_n(t))$ converges. Moreover, by (AOL) techniques (as in Prelims), the defining recurrence relation implies that $\lim p_n(t) = t^{1/2}$ for each $t \in [0, 1]$.

It remains to prove that convergence is *uniform*. **Dini's Theorem** seen in Prelims Analysis II is exactly what we need. For completeness we include a proof for the case we want (more topological than that given in Prelims course notes).

First note that $(f_n(t))$, where $f_n(t) = t^{1/2} - p_n(t)$, is a monotonic decreasing sequence of continuous functions which converges pointwise to 0. Given $\varepsilon > 0$, let

$$F_n = \{\, s \in [0, 1] \mid f_n(s) \geqslant \varepsilon \,\}.$$

Then $F_n$ is closed and $F_{n+1} \subseteq F_n$ for all $n$. Since $[0,1]$ is compact, the Finite Intersection Property implies that if all $F_n$'s were non-empty then $\bigcap_n F_n \neq \emptyset$. But this would imply that there is some point $t$ at which $f_n(t) \not\to 0$, contrary to assumption. So there exists $N$ such that $F_N = \emptyset$. Then $0 \leqslant f_n(s) < \varepsilon$ for all $s \in [0,1]$ and for all $n \geqslant N$. □

It is clear that the set of real polynomials on a closed bounded interval of $\mathbb{R}$ is not closed under the lattice operations of max and min: just consider $t$ and $-t$ on $[-1,1]$. But the polynomials are closed under forming products.

5.8. **Subalgebras of $C(K)$.**

We say that a subspace $A$ of $C(K)$ is a **subalgebra** if $f, g \in A$ implies $fg \in A$ and $A$ contains the constant functions. It is an elementary exercise to show that $A$ a subalgebra implies $\overline{A}$ is a subalgebra too. We claimed earlier that the closure of a subspace is a subspace. So we only need to check that $\overline{A}$ is closed under products. But this comes from a normed space version of the (AOL) result about limits of products of sequences.

Our reason for bringing in closures of subalgebras is revealed by the following proposition.

5.9. **Proposition.** *Let $A$ be a closed subalgebra of $C(K)$. Then $A$ is a linear sublattice.*

*Proof.* It will suffice to show that $f \in A$ implies $|f| \in A$. Assume first that $\|f\| \leqslant 1$. Then $0 \leqslant f^2 \leqslant 1$ and by 5.7 we can find a sequence $(p_n)$ of polynomials such that $p_n(f^2) \to \sqrt{f^2} = |f|$ uniformly on $K$. But $p_n(f^2) \in A$ since $A$ is a subalgebra, and $A$ is closed under uniform limits since $A$ is closed for the sup norm.

For the general case, take $f \neq 0$ and consider $f/\|f\|$, which has norm 1. There is nothing to prove if $f = 0$. □

5.10. **Stone–Weierstrass Theorem (subalgebras form, real case).** *Let $A \subseteq C(K)$ be such that*

(i) *$A$ is a subalgebra of $C(K)$ (that is, it is a subspace closed under products and it contains the constants);*
(ii) *$A$ separates the points of $K$.*

*Then $A$ is dense in $C(K)$.*

*Proof.* By 5.8, $L := \overline{A}$ is a closed subalgebra. By 5.9, $L$ is a linear sublattice. Also $L$ contains constants and $L$ separates points since $A$ does. Now SWT for linear sublattices implies that $L$ is dense. But since $L$ is closed, it equals $C(K)$. □

5.11. **Corollary (Weierstrass's polynomial approximation theorem (real case)).** *Every real-valued continuous function on a closed bounded subinterval of $\mathbb{R}$, or more generally on a compact subset of $\mathbb{R}^n$, is the uniform limit of polynomials.*

5.12. **Example (on Weierstrass's Theorem).**

Suppose that $f \in C[0,1]$ is such that $f$ is real-valued and

$$\int_0^1 t^n f(t)\,\mathrm{d}t = 0 \quad \text{for all } n = 0, 1, 2, \ldots.$$

Then $f$ is identically zero.

*Proof.* Take $\varepsilon > 0$. Choose a real polynomial $p$ such that $\|f - p\| < \varepsilon$. Then, by the assumption, and properties of integration,

$$0 \leqslant \int_0^1 f^2(t)\,\mathrm{d}t = \int_0^1 f(t)(f(t) - p(t))\,\mathrm{d}t.$$

Now

$$\left|\int_0^1 f(t)(f(t) - p(t))\,\mathrm{d}t\right| \leqslant \int_0^1 |f(t)(f(t) - p(t))|\,\mathrm{d}t \leqslant \|f\|\,\|f - p\| \leqslant \|f\|\varepsilon.$$

Since $\varepsilon$ was arbitrary this forces $\int_0^1 f^2(t)\,\mathrm{d}t = 0$. Since $f^2$ is continuous and non-negative, $f^2$, and so also $f$, is identically zero. (Recall 1.11(a).)

5.13. **Concluding remarks on SWT.**

Note that there exist various direct proofs of Weierstrass's polynomial approximation theorem for continuous functions on a closed bounded interval, some of which work for complex-valued functions too. Given Weierstrass's Theorem, one could of course appeal to this to get the general SWT for subalgebras instead of making use of the very special case of it we obtained in 5.7.

The Stone–Weierstrass Theorem, in its subalgebras form, can be extended to the case of complex-valued functions. This extension is not entirely routine and FA-I does not cover it.

5.14. **Separability: introductory remarks.**

We may think of finite-dimensional normed spaces as being 'small' and infinite-dimensional ones as 'large'. But can we distinguish some infinite-dimensional normed spaces as being smaller than others in a way that is meaningful and useful? Certainly a dichotomy based on the number of vectors—countable or uncountable—wouldn't be helpful, since the only countable normed space is $\{0\}$ because the scalar field is uncountable. So what about something more geared to topology?

5.15. **Definition: separable.**

A normed space $X$ is **separable** if there is a countable subset $D$ of $X$ such that $D$ is dense, that is, $\overline{D} = X$. A space which is not separable is said to be **inseparable**.

Explicitly, $D$ is dense in $X$ iff for each $x \in X$,

$$\forall \varepsilon > 0 \;\; \exists s \in D \;\; \|x - s\| < \varepsilon.$$

Of course here $s$ will depend on $x$.

The definition of density in terms of closure makes sense in any metric space (or any topological space), but we focus on normed spaces, in which examples proliferate and in which there are useful results making specific use of the norm.

We record as a lemma an elementary fact which can help make some separability proofs less messy than they might otherwise be.

5.16. **Lemma.** *Let $(X, \|\cdot\|)$ and $(X, \|\cdot\|')$ be normed spaces with equivalent norms. Then either both are separable or both are inseparable.*

*Proof.* A subset $D$ of $X$ is dense in both spaces or neither: either use the definition of equivalent norms, or use the result that the two spaces have the same open sets. □

*It goes without saying that to understand and to work with separability you need to know the rudiments of the theory of countable and uncountable sets, as introduced in. Prelims Analysis I. This will be assumed in this section but a brief summary is provided in a supplementary note, Review of basic facts about countability. It will be legitimate to quote the results recalled there. We enclose such justifications within square brackets in the proofs we give below.*

Let's begin with an example to show that separability can't be taken for granted.

5.17. **Example: $\ell^\infty$ is inseparable.**

We argue by contradiction. Suppose $D$ were a countable dense subset of $\ell^\infty$.

Consider the subset $U$ of $\ell^\infty$ consisting of sequences $(u_j)$ such that $u_j = 0$ or 1 for each $j$. Then $U$ is uncountable [standard example].

Take any 2 elements $u = (u_j)$ and $v = (v_j)$ in $U$ with $u \neq v$. Then there exists $j$ such that $|u_j - v_j| = 1$ so $\|u - v\| \geqslant 1$ (in fact we have equality). Since $D$ is dense, there exist $s_u$ and $s_v$ in $D$ such that

$$\|u - s_u\| < \tfrac{1}{2} \quad \text{and} \quad \|v - s_v\| < \tfrac{1}{2}.$$

If it were the case that $s_u = s_v$ we would have

$$\|u - v\| \leqslant \|u - s_u\| + \|s_u - s_v\| + \|s_v - v\| < 1,$$

which is false. Therefore the map $u \mapsto s_u$ is injective, which is impossible since $U$ is uncountable and $D$ was assumed to be countable [obvious standard fact].

Other examples of inseparable spaces are the space of Lipschitz functions and $L^\infty(\mathbb{R})$. See Problem sheet Q. 17 for details.

5.18. **Proposition (first examples of separable spaces).** *Let the scalar field $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$. The following are separable:*

(1) $\mathbb{F}$;
(2) $\mathbb{F}^m$, *with any norm;*
(3) $\ell^1$.

*Proof.*

(1) $\mathbb{Q}$ is countable and dense in $\mathbb{R}$. $\mathbb{Q} + \mathrm{i}\mathbb{Q} := \{\, a + \mathrm{i}b \mid a, b \in \mathbb{Q} \,\}$ is countable and dense in $\mathbb{C}$ (for, for example, the $\|\cdot\|_\infty$- norm, and hence for any norm by 5.16). So $\mathbb{F}$ has a countable dense subset, henceforth denoted $\mathcal{C}_\mathbb{F}$.

(2) Let's consider $\mathbb{F}^m$ with the $\infty$-norm and let $\mathcal{C}_\mathbb{F}$ be countable and dense in $\mathbb{F}$. Then $(\mathcal{C}_\mathbb{F})^m$ is countable [finite product of countable sets]. Take any $(x_1, \ldots, x_m) \in \mathbb{F}^m$ and any $\varepsilon > 0$. Then for each $j$ there exists $s_j \in \mathcal{C}_\mathbb{F}$ such that $|x_j - s_j| < \varepsilon$. Hence

$$\|(x_1, \ldots, x_m) - (s_1, \ldots, s_m)\|_\infty = \max_j |x_j - s_j| < \varepsilon.$$

Therefore $(\mathcal{C}_\mathbb{F})^m$ is a dense subset of $\mathbb{F}^m$.

Now recall that any two norms on $\mathbb{F}^m$ are equivalent (Theorem 4.3) and appeal to Lemma 5.16.

(3) For $n \geqslant 1$ let

$$D_n = \{(a_1, \ldots, a_n, 0, 0, \ldots)| \mid a_j \in \mathcal{C}_\mathbb{F} \,\}.$$

Then each $D_n$ is countable [finite product of countable sets] and so $D := \bigcup_n D_n$ is countable [countable union of countable sets].

We claim $D$ is dense in $\ell^1$. Let $x = (x_j)$ belong to $\ell^1$ and fix $\varepsilon > 0$. Since $x \in \ell^1$, we can choose $N$ such that $\sum_{n>N} |x_n| < \varepsilon$. So

$$\|x - x^{(N)}\| = \sum_{n>N} |x_n| < \varepsilon, \quad \text{where } x^{(N)} = (x_1, \dots, x_N, 0, 0, \dots).$$

By (2) we can choose $(a_1, \dots, a_N) \in (\mathcal{C}_{\mathbb{F}})^N$ such that $\|(x_1, \dots, x_N) - (a_1, \dots, a_N)\|_1 < \varepsilon$. Let $s = (a_1, \dots, a_N, 0, 0, \dots)$. Then $s \in D$ and $\|x - s\| < 2\varepsilon$. $\square$

These examples illustrate techniques which are available more widely. As ever, the **span** of a set $S$ in a vector space $X$ is

$$\operatorname{span}(S) := \Big\{ \sum_{i=1}^{k} \lambda_i s_i \mid k = 1, 2, \dots, \ \lambda_i \text{ scalar}, \ s_i \in S \Big\}.$$

Crucially, we are only allowed to form *finite* linear combinations here. There is no presumption here in general that $S$ is countable.

The import of the following theorem is, loosely, that we can build in the separability of the scalars and facts about countable sets to arrive at a viable test for separability which avoids the need for messy approximation arguments.

5.19. **Theorem** (testing for separability).

(i) *Let $Y$ be a normed space and let $S$ be a countable set such that* $\operatorname{span}(S) = Y$. *Then $Y$ is separable.*
(ii) *Let $X$ be a normed space and such that* $\operatorname{span}(S)$ *is dense in $X$, where $S$ is countable. Then $X$ is separable.*
(iii) *Let $X$ be a normed space and let $S$ be countable and such that* $\operatorname{span}(\overline{S})$ *is dense in $X$. Then $X$ is separable.*

*Proof.* (i) Every element of $Y$ is a *finite* linear combination of elements of $S$. The set $T$ of finite linear combinations of elements from $S$ with scalars drawn from the countable set $\mathcal{C}_{\mathbb{F}}$ is countable:

$$T = \bigcup_{n \geqslant 1} \Big\{ \sum_{j=1}^{n} a_j s_j \mid a_j \in \mathcal{C}_{\mathbb{F}}, \ s_j \in S \Big\}$$

[countable union of countable sets]. Moreover $T$ is dense in $Y$ (either slog it out or exploit the fact that addition and scalar multiplication are continuous (see 0.8)).

(ii) By (i), applied with $Y = \operatorname{span}(S)$, we see that $Y$ is separable. Let $D$ be countable and dense in $Y$. Then, using superscripts to indicate the space with respect to which the closure is taken.

$$\overline{D}^Y = Y \text{ and } \overline{Y}^X = X.$$

But (by elementary topology),

$$\overline{D}^Y = \overline{D}^X \cap Y.$$

so $Y \subseteq \overline{D}^X$ and, since $\overline{Y} = X$ by assumption it follows that $D$ is dense in $X$. $\square$

(iii) The facts underlying the proof here are that, in a normed space $X$, the span of a subset $P$ is the smallest subspace containing $P$ and that the closure of a subset $Q$ is the smallest closed set containing $Q$.

We have

$$\begin{aligned} S \subseteq \operatorname{span}(S) &\Longrightarrow \overline{S} \subseteq \overline{\operatorname{span}(S)} \\ &\Longrightarrow \operatorname{span}(\overline{S}) \subseteq \overline{\operatorname{span}(S)} \qquad \text{(closure of a subspace is a subspace)} \end{aligned}$$

$$\Longrightarrow \overline{\operatorname{span}(\overline{S})} \subseteq \overline{\operatorname{span}(S)} \qquad \text{(property of closure).}$$

Also

$$S \subseteq \overline{S} \Longrightarrow \operatorname{span}(S) \subseteq \operatorname{span}(\overline{S})$$
$$\Longrightarrow \overline{\operatorname{span}(S)} \subseteq \overline{\operatorname{span}(\overline{S})}.$$

Putting the inclusions together, we get $\overline{\operatorname{span}(\overline{S})} = \overline{\operatorname{span}(S)}$. Hence $X$ is separable, by (ii). □

5.20. **Applications: proofs that particular spaces are separable.**

We first revisit our initial examples of separable spaces from 5.18 and then add some new ones.

(1) $\mathbb{F}^m$ has a countable spanning set so is covered by Theorem 5.19(i).

(2) $\ell^p$ is separable for $1 \leqslant p < \infty$.
This follows from Theorem 5.19(ii) and the fact that the 'coordinate vectors' $e_n = (\delta_{nj})$ form a countable set whose span is dense.

(3) Easy exercise: prove $c_0$ is separable. A slightly less easy result is that $c$, the space of convergent sequences, is separable; see Problem sheet Q. 18.

We now exploit earlier density results by showing that we can find suitable countable subsets $S$ such that $\operatorname{span}(S)$ is dense or $\operatorname{span}(\overline{S})$ is dense.

(4) $C[a, b]$ (real-valued continuous functions on $[a, b] \subseteq \mathbb{R}$, with sup norm) is separable. More generally, $C(K)$, for $K$ a compact subset of $\mathbb{R}^n$, is separable.

*Proof.* For $C[a, b]$: Take $S$ to be the set $\{1, x, x^2, \ldots\}$. Its span is the subspace of all polynomials, and this is dense by Weierstrass's Theorem. For the general result, take $S$ to the set of monomials in variables $x_1, \ldots, x_n$, that is, expressions $x_1^{q_1} \cdots x_n^{q_n}$, where each $q_i \in \mathbb{N}$; note that $S$ is countable. SWT (subalgebra form) implies $\operatorname{span}(S)$ is dense. □

(5) $L^p(\mathbb{R})$ $(1 \leqslant p < \infty)$ is separable.

*Proof.* Apply Theorem 5.19 with $S$ as the set of characteristic functions of bounded intervals with rational endpoints. Here the closure of $S$ in the $L^p$ norm contains all characteristic functions of bounded intervals. Then $\operatorname{span}(\overline{S})$ contains the step functions (why?) and so is dense. [Assumed fact: the step functions are dense in $L^p(\mathbb{R})$.] □

It is a salutary exercise to attempt to prove these results directly from the definition of separability. For example, you might try to show that a suitable countable set of piecewise linear functions is dense in $C[a, b]$. It's messy! Not recommended.

Likewise, direct construction of a countable dense subset in an $L^p$ space is messy.

Now for another general result, which we present in the setting of metric spaces.

5.21. **Theorem.** *Let $Y$ be a subset of a separable metric space $(X, d)$. Then $(Y, d)$ is separable.*

*Proof.* Note that we need to find a countable dense subset *in* $Y$.

Let $D_X = \{x_k\}_{k \in \mathbb{N}}$ be a countable dense set in $X$. Then for each $k$, there exist points $y_{k,j}$ $(j = 1, 2, \ldots)$ in $Y$ such that

$$d(y_{k,j}, x_k) < \operatorname{dist}(x_k, Y) + \frac{1}{j};$$

this comes just from the definition of the distance of a point from a subset in a metric space.

The set $D_Y := \{\, y_{k,j} \mid k, j \in \mathbb{N} \,\}$ is countable [$\mathbb{N} \times \mathbb{N}$ is countable]. We claim $D_Y$ is dense in $Y$. This needs an $\varepsilon/3$-argument. Take $\varepsilon > 0$ and $y \in Y$. By density of $D_X$ in $X$ we can find $x_k$ such that $d(y, x_k) < \varepsilon/3$. This implies that $\operatorname{dist}(x_k, Y) < \varepsilon/3$. Now there exists $j$ such that $j^{-1} < \varepsilon/3$. Then

$$\begin{aligned} d(y, y_{k,j}) &\leqslant d(y, x_k) + d(x_k, y_{k,j}) \\ &\leqslant \frac{\varepsilon}{3} + \frac{1}{j} + \operatorname{dist}(x_k, Y) \\ &\leqslant \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Hence $D_Y$ is dense in $Y$, as claimed. □

5.22. **Separable subspaces: examples and remarks.**

(1) $C_c(\mathbb{R})$ (continuous real-valued functions of compact support, sup norm) is separable.

*Proof.* Use the fact that $C_c(\mathbb{R}) = \bigcup_n C_c^{(n)}(\mathbb{R})$, where $C_c^{(n)}(\mathbb{R})$ is the space of continuous functions on $\mathbb{R}$ which vanish outside $[-n, n]$. Each $C_c^{(n)}(\mathbb{R})$ has a countable dense set, $D_n$ say, since it can be identified with a subset of $C[-n, n]$, which is separable. (It is necessary to say 'a subset of' here. Why?) Then $\bigcup_n D_n$ is dense in $C_c(\mathbb{R})$. □

(2) Observe that we can't get much mileage out of Theorem 5.21 for simplifying our arguments in 5.20 that particular spaces are separable, since we have few instances where the theorem would apply. Certainly $c_0$, with its $\|\cdot\|_\infty$-norm, is a subspace of $\ell^\infty$. Since the latter is inseparable, this does not tell us whether $c_0$ is separable or not. On the other hand, once $c$ is known to be separable, then separability of $c_0$ would follow.

(3) When we apply Theorem 5.21 with the metric $d$ coming from a norm, then the subspace $Y$ is required to carry the norm inherited from $X$. For example, although $\ell^1 \subseteq \ell^2$ (as sets), we could not deduce separability of $(\ell^1, \|\cdot\|_1)$ from separability of $(\ell^2, \|\cdot\|_2)$ because the norms are not the same.

5.23. **Addendum: bases in separable normed spaces?**

It might be tempting to think that in separable normed spaces one could develop a notion of bases, where a 'basis' would be a countable set of vectors forming a linearly independent spanning set. But this is highly problematic. We certainly have no assurance that sums of the form $\sum_{n=1}^\infty \lambda_n x_n$ will be norm-convergent. There is a notion of basis (called a **Hamel basis**) available in an arbitrary vector space $V$, whereby a set $S$ is a spanning set for $V$ if every $v \in V$ is a *finite* linear combination of elements of $S$ and $S$ is linearly independent if every *finite* subset of $S$ is linearly independent in the usual sense. But this is of little practical use: it can be shown with the aid of the Baire Category theorem from FA-II that a Hamel basis for a Banach space is either finite or uncountable.

There is one very special case in which a separable normed space $X$ does have available a good notion of basis. When $X$ is a separable Hilbert space, the notion of orthonormal basis works well. An orthonormal sequence $(x_n)$ is an **orthonormal basis** if $\langle x, xn \rangle = 0$ for all $n$ implies $x = 0$. This concept is explored in FA-II. It has connections with orthogonal expansions as encountered in certain courses in applied analysis and differential equations. The Projection Theorem, stated in 2.12, underpins the theory.

We do, however, have a simple positive result which holds in any separable normed space.

5.24. **Separable normed spaces: a density lemma.** *Let $X$ be a separable normed space and $Y$ a subspace of $X$. Then there exists a sequence*

$$Y = L_0 \leqslant L_1 \leqslant L_2 \leqslant \cdots$$

*of subspaces of $X$ such that*

$$L_\infty := \bigcup_{n=1}^{\infty} L_n$$

*is a dense subspace of $X$.*

*Proof.* Let $D = \{s_1, s_2, \ldots\}$ be a countable dense subset of $X$. Define $(L_n)$ recursively by

$$L_0 = Y, \qquad L_{n+1} = \operatorname{span}(L_n \cup \{s_{n+1}\}) \quad (n \geqslant 0).$$

It is elementary to show that $L_\infty$, as in the statement of the lemma, is a subspace. It remains to prove $L_\infty$ is dense. First observe that $D \subseteq L_\infty$, since each member of $D$ belongs to some $L_n$. Finally

$$X = \overline{D} \subseteq \overline{L_\infty} \subseteq X.$$

So we get equality throughout. □

We conclude this section with a general theorem on the theme 'sometimes it is good enough to have information on a dense subspace'. We shall exploit this and Lemma 5.24 when presenting a proof of the Hahn–Banach Theorem for separable spaces in the next section. The theorem is also needed in FA-II. Note the crucial requirement that the operator must map into a *Banach space.*

5.25. **Extension Theorem for a bounded operator on a dense subspace.** *Let $Z$ be a dense subspace of a normed space $X$. Assume that $Y$ is a Banach space and let $T \in \mathcal{B}(Z, Y)$. Then there exists a unique $\widetilde{T} \in \mathcal{B}(X, Y)$ such that $\widetilde{T}\restriction_Z = T$. Moreover $\|\widetilde{T}\| = \|T\|$.*

*Proof.* Let $x \in X$. Then there exists a sequence $(z_n)$ in $Z$ such that $\|x - z_n\| \to 0$. Now, because $T$ is bounded and linear,

$$\|Tz_m - Tz_n\| \leqslant \|T\| \, \|z_m - z_n\| \leqslant \|T\| \left( \|z_m - x\| + \|x - z_n\| \right).$$

We deduce that $(Tz_n)$ is a Cauchy sequence in $Y$. This converges to an element $y \in Y$, depending on $x$. Denote it by $\widetilde{T}x$. But we now must confront a possible issue of well-definedness.

Suppose we have a rival sequence $(z'_n)$ in $Z$ which also converges to $x$. We need to show that $\lim Tz_n = \lim Tz'_n$. To this end consider

$$\|Tz_n - Tz'_n\| \leqslant \|T\| \, \|z_n - z'_n\| \leqslant \|T\| \left( \|z_n - x\| + \|x - z'_n\| \right).$$

Since the RHS tends to 0, the limits $\lim Tz_n$ and $\lim Tz'_n$ (which we know exist) are the same. Therefore $\widetilde{T}x$ can be unambiguously defined to be $\lim Tz_n$ where $(z_n)$ is any sequence converging to $x$. In addition we see from this that $\widetilde{T}$ extends $T$ (consider a constant sequence). Uniqueness of the extension also follows because any continuous extension of $T$ to $X$, say $\widetilde{\widetilde{T}}$, must be such that, for $z_n \to x$ as above,

$$\widetilde{\widetilde{T}}x = \lim \widetilde{\widetilde{T}}z_n = \lim T z_n.$$

.

Linearity of $\widetilde{T}$ comes from the continuity of addition and scalar multiplication in a normed space. In more detail, let $x, v \in X$ with approximating sequences $z_n \to x$ and $w_n \to v$, with $z_n, w_n \in Z$ for all $n$. Let $\lambda, \mu \in \mathbb{F}$. Then $\lambda z_n + \mu w_n \to \lambda x + \mu v$. But

$$T(\lambda z_n + \mu w_n) = \lambda T z_n + \mu T w_n \to \lambda \widetilde{T}x + \mu \widetilde{T}v$$

and, by continuity of $T$, we also have $T(\lambda z_n + \mu w_n) \to \widetilde{T}(\lambda x + \mu v)$. By uniqueness of limits in a normed space we see that $\widetilde{T}$ is indeed a linear map from $X$ into $Y$.

Moreover, by continuity of norm,

$$\|\widetilde{T}x\| = \|\lim T z_n\| = \lim \|T z_n\| \leqslant \lim \|T\| \, \|z_n\| = \|T\| \, \|x\|.$$

Hence $\widetilde{T}$ is bounded, with norm $\leqslant \|T\|$. But because $\widetilde{T}$ extends $T$ we must have the reverse inequality too, so we get equality. $\square$

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## 6. Dual spaces and the Hahn–Banach Theorem

*In which we present the Hahn–Banach Theorem and give the proof for a separable normed space. In which too we begin to reveal the HBT's powerful and far-reaching consequences.*

### 6.1. **The dual space of a normed space.**

Let $X$ be any normed space. We define the **dual space** of $X$ to be

$$X^* := \mathcal{B}(X, \mathbb{F});$$

its elements are the continuous (*alias* bounded) linear functionals on $X$, equipped with the operator norm

$$\|f\| := \sup\Big\{ \frac{|f(x)|}{\|x\|} \mid x \neq 0 \Big\}$$

(the (Op1) formula used here; any of (Op2), (Op3), (Op4) is available instead, as you prefer). We shall repeatedly use the fact that, for any $f \in X^*$,

$$|f(x)| \leqslant \|f\| \, \|x\| \qquad \text{for all } x \in X.$$

Recall that $X^*$ is always a Banach space, whether or not $X$ is complete (see Corollary 3.9): completeness of $X^*$ comes from the completeness of the scalar field. Warning: choice of notation ($X^*$ or $X'$) for bounded linear functionals is not uniform across the literature and past exam questions.

We slot in here an easy consequence of our Extension Theorem for bounded linear operators, Theorem 5.25. It tells us that a normed space and any dense subspace of it have essentially the same dual space.

6.2. **Proposition.** [dual space of a dense subspace] *Suppose $Z$ is a dense subspace of a normed space $X$. Then the map $J\colon X^* \to Z^*$ given by $Jf = f\restriction_Z$ is a (vector space) isomorphism which is isometric ($\|Jf\| = \|f\|$ for all $f \in X^*$).*

*Proof.* Certainly $J$ is well-defined, linear and continuous. Theorem 5.25 ensures that $J$ is surjective and an isometry. □

We now give a result about linear functionals which is not a specialisation of one true for linear operators.

6.3. **Proposition.** *Let $X$ be a normed space and $f$ a linear functional on $X$.*

(i) *Assume $f \neq 0$. Then there exists $x_0 \in X$ such that*

$$X = \operatorname{span}(\ker f \cup \{x_0\}).$$

*Here $x_0$ may be chosen such that $f(x_0) = 1$.*

(ii) *$f$ is bounded if and only if $\ker f$ is closed.*

*Proof.* (i) Take $x_0$ with $f(x_0) \neq 0$. For any $x \in X$,

$$x - \frac{f(x)}{f(x_0)} x_0 \in \ker f.$$

Hence $X = \operatorname{span}(\ker f \cup \{x_0\})$. To obtain the final assertion note that the span is unchanged if we replace $x_0$ by $x_0/f(x_0)$ and that $f\left(x_0/f(x_0)\right) = 1$.

(ii), $\Longrightarrow$ direction: $\ker f = f^{-1}(\{0\})$, and this is closed if $f$ is continuous.

$\Longleftarrow$ direction: Result is trivial if $f = 0$. So assume $f \neq 0$. Take $x_0$ as in (i), where we may assume that $f(x_0) = 1$. Denote $\ker f$ by $Z$. Let $x \notin Z$. Because $Z$ is closed,

$$0 < \operatorname{dist}(x_0, Z) := \inf\{\, \|x_0 - z\| \mid z \in Z \,\}.$$

Then $x - f(x)x_0 \in Z$ so $x/f(x) - x_0 \in Z$ too. Hence

$$\left\| \frac{x}{f(x)} \right\| \geqslant \delta := \operatorname{dist}(x_0, Z).$$

Hence $|f(x)| \leqslant \delta^{-1}\|x\|$ for all $x \notin Z$. This inequality holds, trivially, for $x \in Z$ too. Hence $f$ is bounded with $\|f\| \leqslant \delta^{-1}$. □

6.4. **Cautionary examples.**

(1) Example of an unbounded linear operator with closed kernel. Let $X := \operatorname{span}\{e_n\}_{n\geqslant 1}$ in $\ell^1$, where as usual $e_n = (\delta_{nj})$. Then $X$ is a dense and proper subspace of $\ell^1$. Define $T\colon X \to \ell^1$ by $T(x_j) = (jx_j)$. Then $\|e_n\|_1 = 1$ and $\|Te_n\|_1 = n$, so $T$ is unbounded. Clearly $\ker T = \{0\}$.

(2) Example of an unbounded linear functional. Let $X$ be the subspace of $C[0,1]$ (real-valued functions, sup norm) such that $X$ consists of the continuously differentiable functions. Let $f\colon X \to \mathbb{R}$ be given by $f(x) = x'(1/2)$ for $x \in X$. Then $f$ is linear but not bounded.

6.5. **Hahn–Banach Theorem: context.**

At its heart, the Hahn–Banach Theorem asserts the existence on normed spaces of bounded linear functionals with particular properties.

The theorem comes in many formulations and FA-I only covers the most basic ones and requires proofs only for separable spaces. The HBT is not a theorem about separable spaces. But its proof is more elementary in that case. [Comments in 6.13 on using set theory machinery (Zorn's Lemma) to remove the separability restriction.]

The proof of the core form of the HBT (6.6) involves inequalities between scalar-valued functions, and so is available only for *real* normed spaces. However, as we are able to show quite easily, the statement of the theorem remains valid for *complex* normed spaces too (see 6.7). Virtually all the spin-off applications we shall give are valid for real or complex normed spaces.

There are some versions of HBT with a geometric flavour and these won't extend to complex normed spaces. We give only hints in FA-I of such theorems (see 6.15). They are however important in more advanced courses, including those focusing on functional analytic methods for PDE's.

6.6. **HAHN–BANACH THEOREM.** *Let $Y$ be a subspace of a real normed space $X$. Let $g \in Y^*$. Then there exists $f \in X^*$ such that*

$$f{\restriction}_Y = g \quad \text{and} \quad \|f\|_X = \|g\|_Y.$$

Thus the bounded linear functional $g$ on $Y$ has an extension to a bounded linear functional $f$ on the whole of $X$, with the same norm. There is no requirement that $Y$ be closed or that $X$ be a Banach space.

*Proof* [for the case that $X$ is separable]

**HBT proof, stage 1.** [Separability used here.] Lemma 5.24 allows us to construct an increasing sequence of subspaces $(L_n)$ with $L_0 = Y$ and the subspace $L_\infty := \bigcup L_n$ dense.

We shall extend $g$ step by step to obtain a norm-preserving extension of $g$ to $L_\infty$—the central part of the proof. This depends on a key lemma.

**HBT proof, stage 2: One-step Extension Lemma.** [The proof here relies crucially on the scalars being $\mathbb{R}$.] *Let $Z$ be a subspace of a real normed space $X$ and let $g \in Z^* = \mathcal{B}(Z, \mathbb{R})$. Let $W := \operatorname{span}(Z \cup \{y\})$ where $y \notin Z$. Then there exists $h \in W^*$ such that $h{\restriction}_Z = g$ and $\|h\|_W = \|g\|_Z$.*

**Proof.** We may assume wlog that $\|g\|_Z = 1$. Every element of $W$ is uniquely expressible as

$$w = z + \lambda y \quad \text{for some } z \in Z,\ \lambda \in \mathbb{R}.$$

If $h \colon W \to \mathbb{R}$ is any linear functional extending $g$ then

$$h(w) = g(z) + \lambda h(y).$$

Our task is to show that we can choose the value of $c := h(y)$ to make $h$ bounded, with norm 1. We need in particular

$$-\|z + y\| \leqslant g(z) + c \text{ and } g(z) + c \leqslant \|z + y\|.$$

Equivalently $c$ has to satisfy

$$(\dagger) \qquad -\|z + y\| - g(z) \leqslant c \leqslant \|z + y\| - g(z).$$

Now for $z, z' \in Z$,

$$g(z) - g(z') = g(z - z') \leqslant \|z - z'\| = \|(z + y) - (z' + y)\| \leqslant \|z + y\| + \|z' + y\|.$$

Hence

$$\alpha := \sup_{z' \in Z} \{-g(z') - \|z' + y\|\} \leqslant \inf_{z \in Z} \{-g(z) + \|z + y\|\} := \beta.$$

We then choose $c \in [\alpha, \beta]$. Then, working backwards, we see that our requirements in (†) are satisfied. But so far we've not considered what $h$ will do to a general element of $W \setminus Z$. We shall now confirm that $\|h\| \leqslant 1$ when $c$ is chosen as above. For $\lambda \neq 0$,

$$|h(z + \lambda c)| = |\lambda||g(z/\lambda) + c| \leqslant |\lambda|\|(z/\lambda) + y\| = \|z + \lambda y\|;$$

the idea here is to exploit the fact that $Z$ is closed under scalar multiplication: we call on (†) with $z/\lambda$ in place of $z$, considering separately the cases $\lambda > 0$ and $\lambda < 0$ to arrive at the required inequality. Finally note that $h$, as an extension of $g$, has $\|h\| \geqslant \|g\|$. $\square$

**HBT proof, stage 3: Extension of $g$ defined on $L_0$ to $k$ defined on $V := L_\infty$.**

By applying the One-step Extension Lemma a finite number of times we can extend $g$ to $h_N \in \mathcal{B}(L_N, \mathbb{R})$ for any finite $N$, where $\|h_N\| = 1$. (It may happen that $L_{m+1} = L_m$ for certain values of $m$. No extension is needed to pass from $L_m$ to $L_{m+1}$ in such cases.) Since $N$ is arbitrary we can see that this process gives us an a well-defined extension $k$ to $V$ for which $\|k\|_V = 1$.

**HBT proof, stage 4: The last lap: exploit density of $V := L_\infty$:**

We can call on our Extension Theorem 5.25 to extend $k \in \mathcal{B}(V, \mathbb{R})$ to $f \in \mathcal{B}(X, \mathbb{R})$ with $\|f\|_X = \|k\|_V = \|g\|_Y$.

6.7. **Hahn–Banach Theorem, complex case.** *Let $X$ be a complex normed space. Let $Y$ be a subspace of $X$ and let $g \in Y^*$. Then there exists $f \in X^*$ such that $f{\restriction}_Y = g$ and $\|f\|_X = \|g\|_Y$.*

*Proof.* We first consider real and imaginary parts of linear functionals on a complex vector space $X$. Let $f$ be a $\mathbb{C}$-linear map from $X$ into $\mathbb{C}$. Write $f(x) = u(x) + \mathrm{i}v(x)$ for each $x \in X$. Then (just calculate) $u$ and $v$ are real-linear and $v(x) = -u(\mathrm{i}x)$ and

$$f(x) = u(x) - \mathrm{i}u(\mathrm{i}x).$$

Conversely, given an $\mathbb{R}$-linear functional $u\colon X \to \mathbb{R}$, we can define $f$ in terms of $u$ as above to obtain a $\mathbb{C}$-linear functional $f\colon X \to \mathbb{C}$ with $\mathrm{re}\, f = u$.

Now consider a complex normed space $X$, subspace $Y$ and a complex-linear bounded linear functional $g$ on $Y$. Restricting to real scalars, regard $X$ as a real space $X_\mathbb{R}$ with $Y$ as a real subspace $Y_\mathbb{R}$, and let $u = \mathrm{re}\, g$. Apply Theorem 6.6 to extend $u$ from $Y_\mathbb{R}$ to a linear functional $w$ on $X_\mathbb{R}$ with $\|u\|_{Y_\mathbb{R}} = \|w\|_{X_\mathbb{R}}$. Define $f$ by $f(x) = w(x) - \mathrm{i}s(\mathrm{i}x)$. This is $\mathbb{C}$-linear and extends $g$. It remains to check that $\|f\|_X = \|g\|_Y$. First of all, $|u(y)| = |\mathrm{re}\, g(y)| \leqslant |g(y)|$ for all $y \in Y$. Hence $\|u\|_{Y_\mathbb{R}} \leqslant \|g\|_Y$. Consider $x \in X$. Then $f(x) \in \mathbb{C}$. We can choose $\theta$ such that $|f(x)| = \mathrm{e}^{\mathrm{i}\theta} f(x)$. Then

$$|f(x)| = f(\mathrm{e}^{\mathrm{i}\theta} x) = \mathrm{re}\, f(\mathrm{e}^{\mathrm{i}\theta} x) = u(\mathrm{e}^{\mathrm{i}\theta} x) \leqslant \|u\|_{X_\mathbb{R}} \|\mathrm{e}^{\mathrm{i}\theta} x\| \leqslant \|g\|_Y \, \|x\|.$$

So $\|f\|_X \leqslant \|g\|_Y$ and the reverse inequality is trivial. $\square$

6.8. **A simple example using HBT.**

Let $X$ be a normed space over $\mathbb{F}$. Let $S = \{x_i\}_{i \in I}$ be a subset of $X$ and $\{c_i\}_{i \in I}$ a corresponding set of complex numbers, and let $M$ be a finite non-negative constant. Prove that a necessary and sufficient condition for there to exist a continuous linear functional $f \in X^*$ such that $f(x_i) = c_i$ for all $i \in I$ and $\|f\| \leqslant M$ is that, for every finite subset $J$ of $I$,

$$\left| \sum_{j \in J} \lambda_j c_j \right| \leqslant M \left\| \sum_{j \in J} \lambda_j x_j \right\|$$

for any choice of scalars $\lambda_j$ $(j \in J)$.

**Solution.**

$\Longrightarrow$ : Assume $f$ exists. Then, for any $J$ and any scalars $\lambda_j$,

$$\Big|\sum_{j\in J}\lambda_j c_j\Big| = \Big|\sum_{j\in J}\lambda_j f(x_j)\Big| = \Big|f\Big(\sum_{j\in J}\lambda_j x_j\Big)\Big| \leqslant \|f\| \,\Big\|\sum_{j\in J}\lambda_j (x_j\Big\| \leqslant M \,\Big\|\sum_{j\in J}\lambda_j (x_j\Big\|.$$

$\Longleftarrow$: Let $Y$ be the subspace of $X$ spanned by $S$, so that the elements of $Y$ are all *finite* linear combinations of elements of $S$. "Define" $f$ on $Y$ by

$$f\Big(\sum_{j\in J}\lambda_j x_j\Big) = \sum_{j\in J}\lambda_j c_j.$$

The key point to observe is that the assumed condition implies that $f$ is well defined: express a vector $x \in Y$ as $x = \sum_{j\in J}\lambda_j x_j$ and $x = \sum_{k\in K}\mu_k x_k$, where $J, K$ are finite subsets of $I$. Consider

$$\sum_{j\in J}\lambda_j x_j - \sum_{k\in K}\mu_k x_k$$

and note that this is an element of $Y$ to which the given condition can be applied to show that the two representations of $x$ give the same value for $f(x)$. Moreover $f$ is linear (routine) and bounded, with $\|f\|_Y \leqslant M$. Also by construction each $x_i \in Y$ and $f(x_i) = c_i$ for each $i \in I$. Now apply HBT to extend $f$ from $Y$ to $X$. □

Corollaries of HBT tumble out fast from Theorems 6.6 and 6.7 with little work needed. We state them as results in their own right but they should be seen as direct or indirect consequences of the main theorem. When not specified the scalar field may be either $\mathbb{R}$ or $\mathbb{C}$.

The idea in all cases is to let $Y$ be a subspace containing the vectors about which we want information, to define a bounded linear functional on $Y$ to capture this information, and then to let HBT do the rest.

Our first result tells us that a normed space (other than $\{0\}$) will always have a plentiful supply of non-zero bounded linear functionals.

6.9. **Proposition.** *Let $X$ be a normed space and $0 \neq x_0 \in X$. Then*

(i) *there exists $f \in X^*$ such that*

$$f(x_0) = \|x_0\| \text{ and } \|f\| = 1;$$

(ii) *there exists $f \in X^*$ such that*

$$f(x_0) = 1 \text{ and } \|f\| = 1/\|x_0\|.$$

*Proof.* (i) Define $Y = \operatorname{span}(\{x_0\})$. Define $g$ on $Y$ by

$$g(\lambda x_0) = \lambda\|x_0\| \quad \text{for } \lambda \text{ any scalar.}$$

Then $g$ is linear and

$$\|g\|_Y = \sup_{\lambda\neq 0}\frac{|g(\lambda x_0)|}{\|\lambda x_0\|} = 1.$$

Now obtain the required $f$ by extending $g$ from $Y$ to $X$.

For (ii), either rescale in (i), replacing $x_0$ by $x_0/\|x_0\|$ or more directly take $g$ on $Y = \operatorname{span}(\{x_0\})$ defined by $g(\lambda x_0) = \lambda$ and apply HBT. □

6.10. **Proposition (separating a point from a closed subspace).** *Let $X$ be a normed space. Let $M$ be a proper closed subspace of $X$ and let $x \in X \setminus M$. Then there exists $f \in X^*$ such that*

$$f(x) = 1, \quad f\restriction_M = 0, \quad \|f\| = \frac{1}{\operatorname{dist}(x, M)}.$$

*Proof.* Note that because $M$ is closed and $x \notin M$ necessarily $\operatorname{dist}(x, M) > 0$.

Define $Y = \operatorname{span}(M \cup \{x\})$. Note that each element of $Y$ has a *unique* representation as $y = m + \lambda x$, where $m \in M$ and $\lambda$ is a scalar. Let $g \colon Y \to \mathbb{F}$ be given by

$$g(m + \lambda x) = \lambda \quad (m \in M,\ \lambda \in \mathbb{F}).$$

Then $g$ is well defined and linear and by construction $g(m) = 0$ for all $m \in M$ and $g(x) = 1$. Finally,

$$\begin{aligned}
\|g\| &= \sup_{0 \neq y \in Y} \frac{|g(y)|}{\|y\|} \\
&= \sup_{0 \neq m + \lambda x,\, \lambda \neq 0} \frac{|\lambda|}{\|m + \lambda x\|} \\
&= \sup_{0 \neq m + \lambda x,\, \lambda \neq 0} \frac{1}{\|x + \lambda^{-1} m\|} \\
&= \frac{1}{\inf\limits_{m' \in M} \|x + m'\|} \\
&= \frac{1}{\operatorname{dist}(x, M)}.
\end{aligned}$$

Now apply HBT. □

6.11. **Theorem (density and bounded linear functionals).** *Let $X$ be a normed space and let $S \subseteq X$. Let $M = \overline{\operatorname{span}}(S)$.*

(i) *If there exists $f \in X^*$ such that $f = 0$ on $S$ but $f \not\equiv 0$ then the linear span of $S$ is not dense.*

(ii) *$M = X$ if, for all $f \in X^*$, $f(s) = 0$ for all $s \in S$ implies $f = 0$.*

*Proof.* For any $f \in X^*$,

$$\begin{aligned}
f(s) = 0 \text{ for all } s \in S &\Longrightarrow f = 0 \text{ on } \operatorname{span}(S) &&\text{(since } f \text{ is linear)} \\
&\Longrightarrow f = 0 \text{ on } \overline{\operatorname{span}}(S) &&\text{(since } f \text{ is continuous).}
\end{aligned}$$

Hence the contrapositive of (i) holds.

For (ii): Observe that $M = \overline{\operatorname{span}}(S)$ is a closed subspace. If $M$ is proper than there exists $x \in X \setminus M$. In that situation we can find $f \in X^*$ as in Proposition 6.10 which is zero on $S$ but not identically 0. □

6.12. **Remarks on Theorem 6.11.**

Both parts of the theorem are important. Note that (i)—a test for non-density—does not use HBT. Problem sheet Q. 19 provides an example of non-density established by the strategy in (i).

Part (ii) should be seen as a density theorem. Given a subset $S$ of a normed space $X$ we can find out whether $\operatorname{span}(S)$ is dense by showing that an element $f$ of $X^*$ for which $f(s) = 0$ for all $s \in S$ has to be identically zero. But in order for this to be useful we shall need to describe the dual space of $X$. We address in the next section the problem of giving concrete descriptions for various of our familiar normed spaces and illustrate the use of Theorem 6.11 (further examples in problem sheets).

In subsequent sections we shall also see that there is much more to be said about HBT applications in general. In particular HBT plays a key part in the investigation of duals of bounded linear operators.

**Annexe to Section 6** [optional extras]**: More about the Hahn–Banach Theorem**

Here we stray beyond the narrow confines of the Part B syllabus in two directions.

6.13. **The proof of Theorem 6.6 without the restriction to separable spaces.**

[*This subsection is aimed at those planning to take Part B Set Theory, and at the naturally curious.*]

We need the principle of set theory known as **Zorn's Lemma**. The idea is to extend $g$, a bounded linear functional on $Y$, without change of norm, to a maximal possible domain, and we hope to show that there is an extension to the whole of $X$. Let's consider the set of all possible extensions:

$$\mathcal{E} := \{\, h \in \mathcal{B}(Z, \mathbb{R}) \mid Z \text{ is a subspace of } X \supseteq Y \text{ and } h{\restriction_Y} = g \,\}.$$

Then we can regard $\mathcal{E}$ as being partially ordered (formally one says $h_1 \leqslant h_2$ iff $\operatorname{graph} h_1 \subseteq \operatorname{graph} h_2$, which is a fancy way of saying that $h_2$'s domain contains $h_1$'s domain and that $h_2$ extends $h_1$).

Suppose we have an element of $\mathcal{E}$ which is *maximal*, that is, an extension $f$ of $g$, with domain $Z$ say, which cannot be extended any further. Then EITHER $Z = X$ and we have the extension we were seeking OR $Z \neq X$ and we can apply the One-step Extension Lemma to extend $f$ to $\operatorname{span}(Z \cup \{x\})$, thereby contradicting maximality. But how do we get a maximal element?

> **Zorn's Lemma:** *Let $\mathcal{E}$ be a non-empty family of sets partially ordered by inclusion and assume that $\bigcup_{i \in I} S_i \in \mathcal{E}$ whenever $\{S_i\}_{i \in I}$ is a* **chain** *of elements of $\mathcal{E}$, meaning that for any $i, j \in I$, either $S_i \subseteq S_j$ or $S_j \subseteq S_i$. Then $\mathcal{E}$ has an element which is maximal (with respect to set inclusion).*

The assumption that $\mathcal{E} \neq \emptyset$ is important. In the HBT example the initial functional $g$ on $Y$ gives an element of $\mathcal{E}$.

The idea of a chain of subsets isn't new: our subspaces $\{L_n\}$ constructed in **??** and used in the HBT proof 6.6 form a chain of sets indexed by $\mathbb{N}$ and their union is a subspace to which $g$ naturally extended. For the general case the idea is that extensions belonging to a chain are compatible, allowing us to take their union and get another extension.

Similarly, in algebra the union of a chain of proper ideals in a commutative ring with 1 is a proper ideal; applying Zorn's Lemma we get that every proper ideal is contained in a maximal ideal.

6.14. **Relaxing the assumptions in the statement of the real HBT.**

We used the norm function to measure how big an extension of a given functional $g$ is. But careful inspection of the proof of Theorem 6.6 shows that we didn't make use of all the properties of a norm. With $X$ a real vector space we could have replaced $\|\cdot\|$ by a **sublinear functional** $p\colon X \to \mathbb{R}$ satisfying

$$\begin{aligned} p(x+y) &\leqslant p(x)+p(y) \quad \forall x,y\in X,\\ p(\alpha x) &= \alpha p(x) \quad \forall x\in X,\ \alpha>0. \end{aligned}$$

The conditions may be seen as weakening of norm condition (N2) and abandoning of (N1). Obviously any norm, any linear functional and any seminorm is a sublinear functional.

> **Real HBT, sublinear functional form.** *Let $X$ be real vector space equipped with a sublinear functional $p$. Assume that $Y$ is a subspace of $X$ and that $g\colon Y \to \mathbb{R}$ is lear and such that $|g(y)| \leqslant p(y)$ for all $y \in Y$. Then there exists a linear functional $f$ on $X$ which extends $g$ and is such that $|f(x)| \leqslant p(x)$ for all $x \in X$.*

6.15. **Real Hahn–Banach Theorem: a geometric view.** There is a very extensive literature on geometric forms of the HBT, of which we give only the briefest possible glimpse.

Recall from 6.3 that the kernel of a non-zero linear functional $f$ on $X$ has **codimension** 1, that is, that there exists $x_0$ such that $X = \operatorname{span}(\ker f \cup \{x_0\})$. This means that we may think of a set $\{\, x \in X \mid f(x) = \alpha \,\}$, which is a translate of the subspace $\ker f$, as a **hyperplane**, $H_\alpha$. This may be viewed as dividing $X \setminus \ker f$ into two **half-spaces**, $\{\, x \in X \mid f(x) < \alpha \,\}$ and $\{\, z \in X \mid f(x) > \alpha \,\}$, whose closures intersect in $H_\alpha$ (assuming $f$ is continuous).

One may then think of Proposition 6.9(i) as constructing a hyperplane which is tangent to the closed unit ball at a vector $x_0$ of norm 1.

In the other direction, one may, for example, seek to separate a closed convex set $V$ in a real normed space from a point $x$ not in $V$ by a hyperplane given by an element $f$ of $X^*$ so that $V$ lies in one of the associated open half-spaces and $x$ in the other.

A systematic approach to results of this type (of which there are many variants) one can proceed in a manner similar to that used with a sublinear functional $p$, employing a convex real-valued function $F$ in place of $p$ and replacing the inequality $|g(y)| \leqslant p(y)$ for $y \in Y$ by $g(y) \leqslant F(y)$, and likewise for extensions. But obtaining HBT in the convex functions form and its applications are a specialised topic. The work involved in carrying this through is disproportionate if one's primary interest is in the basic applications of HBT given earlier in this section and in the following ones.

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## 7. Dual spaces of particular spaces; further applications of the Hahn–Banach Theorem

*In which we investigate the dual spaces of some familiar normed spaces and illustrate how these can be exploited in conjunction with the Hahn–Banach Theorem. In addition we apply HBT techniques to obtain various theoretical results.*

7.1. **The dual space of a finite-dimensional vector space.**

Revise your Part A Linear Algebra! Let $X$ be a finite-dimensional vector space over $\mathbb{F}$, where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$. We define $X'$ to be the space of linear functionals (= linear maps from $X$ into $\mathbb{F}$) with addition and scalar multiplication defined pointwise. Then

- for any given basis $\{e_1, \ldots, e_m\}$ of $X$ there exists a **dual basis** $\{e'_1, \ldots, e'_m\}$ for $X'$ with $e'_j(e_i) = \delta_{ij}$, and hence
- $\dim X' = \dim X$ (so $X$ and $X'$ are isomorphic).

Moreover,

- there is a canonical (= basis-free) isomorphism from $X$ onto its second dual $X''$ given by $x \mapsto \varepsilon_x$, where $\varepsilon_x(f) = f(x)$ for all $f \in X'$.

**A key example:** $X = \mathbb{R}^m$. Here, for any $y \in \mathbb{R}^m$ we get a linear functional $f_y \colon x \mapsto x \cdot y$, where $\cdot$ denotes scalar product. Writing $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$ the formula is

$$f_y(x) = \sum_{j=1}^{m} x_j y_j.$$

Taking the standard basis $\{e_1, \ldots, e_m\}$ for $\mathbb{R}^m$, we find that $f_{e_j}(e_i) = \delta_{ij}$, so that the dual basis is $\{f_{e_1}, \ldots, f_{e_m}\}$.

If we identify $f_y$ with $y$ then we may think of $(\mathbb{R}^m)'$ as being $\mathbb{R}^m$. [If one thinks of the vectors $x$ as column vectors and the vectors $y$ as row vectors then $y(x)$ is simply given by matrix multiplication.]

When we make the transition from finite-dimensional spaces (pure Linear Algebra) to normed spaces, we want to consider continuous linear functionals. if $X$ is finite-dimensional then every linear functional on $X$ is automatically continuous, by Theorem **??**. This means that we already know from 7.1 how to describe the elements of $X^*$ and how they act on elements of $X$. The only novel feature is the norm structure, in particular how the chosen norm in $X$ relates to the operator norm of $X^*$. Not for the first time there are parallels between $\ell^p$ spaces and spaces $(\mathbb{F}^m, \|\cdot\|_p)$; see 7.5 below.

7.2. **Dual spaces of particular normed spaces.**

Having an explicit description of the dual space $X^*$ of a given normed space $X$ gives us a lot of valuable information we can exploit in various ways to solve problems involving $X$.

Specifically, given $X$, we would like to find a space $(Y, \|\cdot\|)$ (it'll necessarily be a Banach space) and a map $J \colon Y \to X^*$ such that

(A) $J$ is well defined, that is, $Jy$ is a bounded linear functional on $X$ for each $y \in Y$;
(B) $J$ is linear;
(C) $\|Jy\| \leqslant \|y\|$ for all $y \in Y$, so $J$ is bounded;
(D) $\|Jy\| \geqslant \|y\|$ for all $y \in Y$ (assuming (C) holds, this forces $J$ to be injective);
(E) $J$ is surjective.

Here (C) and (D) together show that $J$ is *isometric*, that is, $\|Jy\| = \|y\|$. Also (C), (D) and (E) together imply that $J^{-1}$ is a well-defined map which is linear and bounded.

When such a $J$ exists we say that $X^*$ is **isometrically isomorphic** to $Y$ and write $X^* \cong Y$. This is the appropriate notion of isomorphism for normed spaces, telling us that the spaces involved can if we wish be identified.

Finding a candidate for $Y$ and verifying (A)–(E) may be fairly easy (sequence spaces, except $\ell^\infty$); may involve sophisticated machinery to carry through in full (spaces of continuous functions, Lebesgue spaces, except $L^\infty$); or the dual space may be unwieldy and really hard to describe ($\ell^\infty$, $L^\infty$). See 7.5 for more explanation.

7.3. **Theorem (dual spaces of sequence spaces).**

(1) $(c_0)^* \cong \ell^1$;
(2) $(\ell^1)^* \cong \ell^\infty$;

(3) $(\ell^2)^* \cong \ell^2$;

(4) $(\ell^p)^* \cong \ell^q$, *for* $1 < p < \infty$, *where* $p^{-1} + q^{-1} = 1$.

*In each of the four cases, the dual space* $X^*$ *of the sequence space* $X$ *is identified via an isometric isomorphism* $J$ *with another sequence space,* $Y$ *say. When this identification is made, the action of elements of* $Y$ *on elements of* $X$ *is given by*

$$(\eta_j)(x_j) = \sum_j x_j\eta_j \qquad \textit{for all } (x_j) \in X,\ (\eta_j) \in Y.$$

*Proof.* To illustrate the strategy we prove (1) in gory detail, following the checklist (A)–(E), and paying particular attention to issues of well-definedness, which are often linked to convergence issues.

(A) **Defining** $J$: Given $x = (x_j) \in c_0$ and $\eta = (\eta_j) \in \ell^1$, "define"

$$(J\eta)(x) = \sum x_j\eta_j.$$

Check:

(i) $(J\eta)(x) \in \mathbb{F}$: we need $\sum x_j\eta_j$ to converge. Proof: $\sum |x_j\eta_j|$ converges by comparison with $\sum |\eta_j|$ since $(x_j)$ is bounded; hence $\sum x_j\eta_j$ converges.

(ii) $J\eta$ is linear for each $\eta$: We need to show $(Jy)(\lambda x+\mu x') = \lambda(J\eta)(x)+\mu(J\eta)(x')$, for $x, x' \in c_0$ and $\lambda, \mu \in \mathbb{F}$. This is routine to check.

(iii) $J\eta$ is bounded for each $\eta$:

$$|(J\eta)(x)| = \left|\sum x_j\eta_j\right| \leqslant \sum |x_j\eta_j| \leqslant \sup_j |x_j| \sum |\eta_j|;$$

here we have used the fact that $\sum |x_j\eta_j|$ converges (from (i)) and the Triangle Inequality for infinite sums of scalars (Prelims exercise). Therefore

$$\|J\eta\| \leqslant \|\eta\|_1.$$

(B) $J$ **is linear**: We need to show that, for $\eta, \eta' \in \ell^1$ and $\lambda, \mu \in \mathbb{F}$,

$$J(\lambda\eta + \mu\eta') = \lambda J\eta + \mu J\eta',$$

that is, that

$$\big(J(\lambda\eta + \mu\eta')\big)(x) = (\lambda J\eta + \mu J\eta')(x) \quad \text{for all } x \in c_0,$$

that is, that

$$\big(J(\lambda\eta + \mu\eta')\big)(x) = \lambda(J\eta)(x) + \mu(J\eta')(x) \quad \text{for all } x \in c_0,$$

because vector space operations in a dual space are defined pointwise. This last equation is straightforward to check. [(B) is routine but clear thinking is required about what needs to be checked.]

(C) **Upper bound on** $\|J\eta\|$: Already proved in (A)(iii).

(D) **Lower bound on** $\|J\eta\|$: We let $0 \neq \eta = (\eta_j) \in \ell^1$ and look at the effect of $J\eta$ on suitably chosen vectors in $c_0$ to get a lower bound for $\|J\eta\|$. For $n = 1, 2, \ldots$, define $x^{(n)} = (x_{nj})_{j\geqslant 1}$ by

$$x_{nj} = \begin{cases} \dfrac{|\eta_j|}{\eta_j} & \text{if } j \leqslant n \text{ and } \eta_j \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that each $x^{(n)} \in c_0$. Moreover $x^{(n)}$ has norm 1 if $x^{(n)} \neq 0$ and this is the case for large enough $n$. Then, for such $n$,

$$\|J\eta\| \geqslant |(J\eta)(x^{(n)})| = \sum_{j=1}^{n} |\eta_j|.$$

It follows that $\|J\eta\| \geqslant \sum_{j=1}^{\infty} |\eta_j| = \|\eta\|$. (There is nothing to do if $\eta = 0$.)

(E) $J$ **surjective**: Take any $f \in (c_0)^*$. Let $\eta = (\eta_j)$ where $\eta_j := f(e_j)$. We claim that $f = J\eta$. This requires two steps.

(i) $(\eta_j) \in \ell^1$: We have to show that $\sum |f(e_j)|$ converges. By Prelims Analysis this will be true provided the partial sums are bounded above. We make use of $x^{(n)}$ as defined in (D) where we now take $y_j = f(e_j)$. Then by linearity of $f$ we have

$$f(x^{(n)}) = f\Big(\sum_{j=1}^{n} x_{nj} e_j\Big) = \sum_{j=1}^{n} x_{nj} f(e_j) = \sum_{j=1}^{n} |\eta_j|.$$

Hence, for all $n$,

$$\|f\| \geqslant |f(x^{(n)})| = \sum_{j=1}^{n} |\eta_j|,$$

as required.

(ii) $(J\eta)(x) = f(x)$ **for any** $x = (x_j) \in c_0$:

$$|f(x) - \sum_{j=1}^{n} x_j \eta_j| = |f(x) - f(\sum_{j=1}^{n} x_j e_j)| \leqslant \|f\| \, \|x - \sum_{j=1}^{n} x_j e_j)\| = \|f\| \sup_{j>n} |x_j|.$$

Since $x \in c_0$ we deduce that

$$f(x) = \sum_{j=1}^{\infty} x_j \eta_j = (J\eta)(x).$$

This completes the proof of (1). The way that elements $\eta$ of $\ell^1$ act as linear functionals on elements $x$ of $c_0$ is immediate from the proof we have given.

The characterisations (2), (3) and (4) are handled likewise, with (A)–(C) requiring almost no adaptation and so routine. Variants are needed for (D) for different spaces, and (E) is then modified accordingly. Check the details for yourself, referring to textbooks for confirmation. □

7.4. **Example: density in $\ell^1$.** (from 2008:b4:3) on use of 6.11

Let $\beta_1, \beta_2, \ldots$ be distinct elements of $\mathbb{C}$ such that there exists $\delta$ such that for $k = 1, 2, \ldots$,

$$|\beta_k| \leqslant \delta < 1.$$

For each $k$ define the sequence $x_k$ by

$$x_k = ((\eta) = (1, \beta_k, \beta_k^2, \ldots).$$

Show that $x_k$ lies in the sequence space $\ell^1$ and that the closed linear span

$$\overline{\operatorname{span}}\{\, x_k \mid k = 1, 2, \ldots \}$$

coincides with $\ell^1$.

**Solution.** Certainly each $x_k \in \ell^1$. Now we seek to apply Theorem 6.11. Suppose that $f \in (\ell^1)^*$ is such that $f(x_k) = 0$ for all $k$. By Theorem **??**(2), we can identify $f$ with $(\eta_j) \in \ell^\infty$ to get

$$0 = \sum_{j \geqslant 1} \beta_k^{j-1} \eta_j \quad \text{for all } k.$$

Reinterpreting this we see that

$$F(z) := \sum_{n=0}^{\infty} \eta_{n+1} z^n$$

takes the value 0 at each $a = \beta_k$. But $F$ is defined by a power series which has radius of convergence at least 1 since $(\eta_{n+1})$ is bounded. It follows that $F$ is holomorphic in the open unit disc. It has an infinite set of zeros in the closed disc $\overline{\mathbb{D}}(0, \alpha)$. By the Bolzano–Weierstrass Theorem, $F$ has a limit point of zeros in the open unit disc. By the Identity Theorem $F \equiv 0$. This implies that $\eta_j = 0$ for all $j$, so $f \equiv 0$. The required result follows from Theorem 6.11(ii). $\square$

7.5. **Further dual space characterisations, and remarks.**

- **Finite-dimensional spaces**: The duals of the spaces $\mathbb{F}^m$ with $p$-norms $(1 \leqslant p < \infty)$ follow exactly the same pattern as do their $\ell^p$ analogues except that convergence checks are not involved in the proofs and surjectivity van be handled by a dimension argument (recall 7.1). All that is new is identification of the dual space norm: $(\mathbb{F}^m, \|\cdot\|_p)^* \cong (\mathbb{F}^m, \|\cdot\|_q)$, where $p^{-1}+q^{-1} = 1$. Moreover we also have $(\mathbb{F}^m, \|\cdot\|_\infty)^* \cong (\mathbb{F}^m, \|\cdot\|_1)$: the convergence obstacles which arise if we try to apply the techniques of Theorem 7.3 to $\ell^\infty$ don't arise.
- **Spaces of continuous functions**: Characterisations of dual spaces go deep into measure theory. But (see Problem sheet Q.??), it is not difficult to show that $C[0,1]^*$ contains a proper dense subspace which is isometrically isomorphic to $L^1[0,1]$.
- $\ell^\infty$: In the proof of the characterisation of $(c_0)^*$ we made use of the fact that we were dealing with sequences which converge to 0, and not with sequences which are merely bounded, in just one place: in (E) (proving surjectivity of $J$). We deduce that $(\ell^\infty)^*$ contains a subspace isometrically isomorphic to $\ell^1$. But maybe it contains much more? This is not an easy question!
- **Lebesgue spaces**: The pattern seen for $\ell^p$ (a special instance of an $L^p$ space) persists: $L^p(\mathbb{R})^*$ can be identified with $L^q(\mathbb{R})$ $(1 < p < \infty,\ p^{-1} + q^{-1} = 1)$ and $L^1(\mathbb{R})^*$ can be identified with $L^\infty(\mathbb{R})$.

  As might be expected, $L^\infty(\mathbb{R})^*$ is large and elusive. Similar claims hold when $\mathbb{R}$ with Lebesgue measure is replaced by other measure spaces. All this goes way beyond FA-I.

7.6. **A special case: dual spaces of Hilbert spaces.**

To avoid distractions keeping track of complex conjugate signs we shall assume in this subsection that the scalars are real. All the results we mention have complex analogues.

Our catalogue of dual spaces in Theorem 7.2(4) includes the result that $(\ell^2)^*$ is isometrically isomorphic to $\ell^2$. Moreover, for $x = (x_j)$ and $y = (y_j)$ in $\ell^2$,

$$\langle x, y \rangle = \sum_{j=1}^{\infty} x_j y_j$$

and the expression on the right-hand side is the same as that we have when we regard ($y$ as an element of $(\ell^2)^*$. In other words, every bounded linear functional on $\ell^2$ is of the form $f_y \colon x \mapsto \langle x, y \rangle$, for some $y \in \ell^2$.

This exactly parallels what you have seen in Linear Algebra for finite-dimensional (real) inner product spaces: the Riesz Representation Theorem. And we can bring the Linear Algebra theorem within the normed spaces framework by equipping $\mathbb{R}^m$ with the Euclidean norm associated with the usual scalar product.

So far we've only considered particular Hilbert spaces. With a sneak preview of FA-II territory, we record the general Riesz Representation Theorem (real case) and make brief comments.

> **Riesz Representation Theorem** *Let $X$ be a real Hilbert space. Then there is a linear isometry $J$ of $X$ onto $X^*$, given by:*
>
> $$(Jy)(x) = \langle x, y\rangle \qquad (x, y \in X).$$

The proof depends on Proposition 6.3 together with, to prove surjectivity of $J$, the Projection Theorem 2.12 applied with $Z = \ker f$ for $f \in X^*$.

It is interesting to review the Hahn–Banach Theorem in the context of Hilbert spaces. Suppose we have a subspace $Y$ of a real Hilbert space $X$ and $g \in Y^*$. The subspace $Y$ is not assumed to be closed. However $Y$ is a dense subspace of $Z := \overline{Y}$. and we can appeal to Proposition 6.2 to extend $g$ to $Z$ without changing the norm. So we may assume without loss of generality that $Y$ is closed. By Proposition **??**, $Y$ is a Hilbert space for the induced IPS norm. Then, by RRT applied to $Y$, there exists $y \in Y$ such that $g = f_y$, where $f_y(x) = \langle x, y\rangle$ for all $x \in Y$. Then $f_y$ is the restriction to $Y$ of the continuous linear functional $x \mapsto \langle x, y\rangle$ on $X$. Both $f_y$ and its extension to $X$ have norm $\|y$
. This proved the HBT for (real) Hilbert spaces. Note that separability is not relevant to this argument.

### 7.7. **Aside: Dual spaces and separability.**

It need not be the case that a separable normed space has a separable dual: consider $\ell^1$.

On the other hand it can be shown that if $X^*$ is separable then $X$ is necessarily separable. [See for example 2011, Paper b4, Question 2 for an outline proof—quite tough.]

### 7.8. **Looking further afield: second duals.**

We mentioned that a finite-dimensional vector space $X$ is naturally isomorphic to its second dual $X''$. We remark here that we cannot expect in general to have $X \cong X^{**}$ for a normed space. In particular this could not occur whenever $X$ is not complete. Also, by way of an example of a different type, $c_0$ is separable, but $(c_0)^{**} \cong (\ell^1)^* \cong \ell^\infty$, which is inseparable and so not isometrically isomorphic to $c_0$.

On the other hand we do have

$$(\ell^p)^{**} \cong (\ell^q)^* \cong \ell^p \qquad (1 < p < \infty,\ p^{-1} + q^{-1} = 1).$$

### 7.9. **Embedding a normed space into its second dual.**

Let $X$ be a normed space. For each $x \in X$, define a map $\varepsilon_x$ by

$$\varepsilon_x(f) = f(x) \quad \text{for all } f \in X^*;$$

think of $\varepsilon_x$ as 'evaluate each $f \in X^*$ at $x$'. Let $J\colon x \mapsto \varepsilon_x$ $(x \in X)$.

**Claims**

(i) $\varepsilon_x \in X^{**}$ for each $x$;
(ii) $J\colon X \to X^{**}$ is well defined and linear;
(iii) $J$ is isometric (and hence injective).

Note that there are two linearity checks required here, both routine, and that the one in (i) is what ensures that $J$ in (ii) is well defined.

Now consider (iiii). We have

$$\begin{aligned} \|Jx\| &= \sup\{\, |(Jx)(f)| \mid f \in X^* \text{ and } \|f\| = 1\,\} \\ &= \sup\{\, |f(x)| \mid f \in X^* \text{ and } \|f\| = 1\,\} \\ &\leqslant \|x\|. \end{aligned}$$

To get the reverse inequality we invoke Proposition 6.9(i). Given $x \in X$ there exists $f \in X^*$ with $f(x) = \|x\|$ and $\|f\| = 1$ and this witnesses that the sup is attained. In fact we have for any $x$ that

$$\|x\| = \sup \| \, |f(x)| \mid f \in X^* \text{ and } \|f\| = 1\,\}.$$

Now let us define

$$\widehat{X} := JX \subseteq X^{**}.$$

This gives a faithful copy of $X$ inside the Banach space $X^{**}$. We say that $X$ is **reflexive** if $JX = X^{**}$. We do not pursue the theory of reflexive spaces in B4.1—this is substantial and belongs in a more advanced course (Part C level).

7.10. **Completion of a normed space.**

Any reflexive space $X$ is automatically a Banach space. But let's consider a general normed space $X$. With notation as above we have a subspace $\widehat{X} = JX$ of the Banach space $X^{**}$ which is isometrically isomorphic to the original space $X$. Consider $\widetilde{X} := \overline{\widehat{X}}$. Then $\widetilde{X}$ is a closed subspace of $X^{**}$ and hence is a Banach space. Moreover $JX$ is dense in $Z$, We have proved that any normed space has a **completion** in which it embeds as a dense subspace. Note we needed the HBT to show that $J$ faithfully embeds $X$ into $\widetilde{X}$, in a form applicable to all normed spaces [real or complex] r

We can say more about our completion $(J, \widetilde{X})$ of $X$. It is characterised by what is known as a **universal mapping property**: given any completion $(i, Z)$ of $X$ (so $i$ is a linear isometry embedding $X$ into a Banach space $Z$) then there exists a linear isometry $\widetilde{i} \colon \widetilde{X} \to Z$ such that the following diagram commutes:

$$\begin{tikzcd} & \widetilde{X} \arrow[d, dashed, "\widetilde{i}"] \\ X \arrow[ur, "J"] \arrow[r, "i"'] & Z \end{tikzcd}$$

The proof of the existence of the lifting $\widetilde{i}$ comes straight from Theorem 6.1.

Compare this construction with that of the completion of a metric space on Problem sheet Q. 7. There we worked with a bigger class of spaces, but the completion we obtained was not a Banach space or the embedding linear, even when the metric came from a norm. However a universal mapping property analogous to that for the normed space completion can be proved for the metric space completion.

## 8. Bounded linear operators: dual operators; spectral theory

*In which we pursue further the theory of bounded linear operators, with more powerful techniques available than we had in Section 3. Specifically we shall exploit the Hahn–Banach Theorem in various ways. One such application deals with the dual of a bounded linear operator, extending the theory from Part A Linear Algebra to the setting of normed spaces.*

*Building on our earlier study of invertible operators we demonstrate that in infinite dimensional spaces there's much more to spectral theory than eigenvalues. We extend the idea of spectrum to normed spaces. We exhibit a number of general results about the spectrum of a bounded linear operator on $X$; the deeper ones require $X$ to be a complex Banach space.*

### 8.1. **Introducing dual operators.**

**Recap from Part A**:

If $X$ and $Y$ are finite-dimensional vector spaces and $T\colon X \to Y$ is a linear map then we can define a linear map $T'\colon Y' \to X'$ such that

$$(T'g)(x) = g(Tx) \quad \text{for all } g \in Y',\ x \in X.$$

Here $X'$ and $Y'$ are the vector spaces of *all* linear functionals on $X$ and $Y$ respectively; $T'\colon g \mapsto g \circ T$ and is certainly a well-defined and linear map. The theory of dual transformations in the finite-dimensional setting focused on the relationships between the kernel (respectively, image) of $T$ and the image (respectively, kernel) of $T'$ and on matrix representations with respect to bases for $X$ and $Y$ and corresponding dual bases for $Y'$ and $X'$. All of this relied heavily on dimension arguments, including the Rank-Nullity Theorem.

The special case of $T\colon X \to X$, where $X$ is a finite-dimensional inner product space received special attention in Prelims and Part A: there the notion of an adjoint $T^*\colon X \to X$ was paramount. A tie-up between $T^*\colon X \to X$ and $T'\colon X' \to X'$ results from identifying $X'$ with $X$ itself using the Riesz Representation Theorem for linear functionals on an inner product space (care needed with complex conjugate signs in the case of complex scalars). [We don't consider in FA-I adjoint operators on Hilbert spaces (= inner product spaces with a Banach space norm): that's a major topic in FA-II.]

What can we do with dual maps in the context of normed spaces? Given $T \in \mathcal{B}(X, Y)$, where $X, Y$ are normed spaces, we can define a map $T'$ by

$$(T'\varphi)(x) = \varphi(Tx) \text{ for all } \varphi \in Y^*,\ x \in X.$$

Paralleling the finite-dimensional case, $T'\colon \varphi \mapsto \varphi \circ T$. Proposition 8.2 confirms that $T'$ is a bounded linear operator from $Y^*$ to $X^*$. There is a notational awkwardness here we cannot avoid. We reserve the notation $X'$ for the space of all linear functionals on any space $X$ and so chose to use $X^*$ for the dual space of a normed space $X$, whose elements are the bounded (*alias* continuous) linear functionals on $X$. We use the notation $T'$ for the map dual to $T$ since $T^*$ is too well-established usage in the restricted setting of inner product spaces to be a sensible choice here.

8.2. **Proposition (dual operator).** *Let $X$, $Y$ be normed spaces (over the same field, $\mathbb{R}$ or $\mathbb{C}$) and $T \in \mathcal{B}(X, Y)$. Then $T' \in \mathcal{B}(Y^*, X^*)$ and $\|T'\| = \|T\|$.*

*Proof.* We only prove that each $T'\varphi$ is bounded and $T'$ is bounded and that $\|T'\| = \|T\|$ (the rest is just linear algebra).

$$|(T'\varphi)(x)| = |\varphi(Tx)| \leqslant \|\varphi\|\,\|Tx\| \leqslant \|\varphi\|\,\|x\|\,\|T\|.$$

Hence $T'\varphi$ is bounded, with $\|T'\varphi\| \leqslant \|T\|\,\|\varphi\|$. Thence $T'$ is also bounded, with $\|T'\| \leqslant \|T\|$.

For the reverse inequality for the norm, we need HBT. Take $x \in X$. Assume first that $Tx \neq 0$. By Proposition 6.9(i) there exists $\varphi \in Y^*$ such that $\varphi(Tx) = \|Tx\|$ and $\|\varphi\| = 1$. Then

$$\|Tx\| = |\varphi(Tx)| = |(T'\varphi)(x)| \leqslant \|T'\varphi\| \, \|x\| \leqslant \|T'\| \, \|x\|,$$

and this also holds, trivially, if $Tx = 0$. Therefore $\|T\| \leqslant \|T'\|$. □

8.3. **Annihilators; kernels and images of bounded linear operators and their duals.**

Let $X$ be a normed vector space. For $S \subseteq X$ and $Q \subseteq X^*$ let

$$S^\circ = \{ f \in X^* \mid f(x) = 0 \text{ for all } x \in S \},$$
$$Q_\circ = \{ x \in X \mid f(x) = 0 \text{ for all } f \in Q \}.$$

Then $S^\circ$ and $Q_\circ$ are closed subspaces of $X^*$ and $X$ respectively (easy exercise). Moreover [from a problem sheet question on HBT], for any subspace $Y$ of $X$,

$$\overline{Y} = (Y^\circ)_\circ :$$

note the closure sign!

This leads on, easily, to the following results. The proofs are left as exercises. Let $X, Y$ be normed spaces and $T \in \mathcal{B}(X, Y)$, with dual operator $T' \in \mathcal{B}(Y^*, X^*)$. Then

$$(TX)^\circ = \ker T', \qquad \overline{TX} = (\ker T')_\circ;$$
$$(T'Y^*)_\circ = \ker T, \qquad \overline{T'Y^*} \subseteq (\ker T)^\circ.$$

Where closure signs appear in the general results, they aren't needed when the domain or the range of the operator in question is finite-dimensional (why?). Moreover, when $T'Y^*$ is finite-dimensional, $T'Y^* = (\ker T)^\circ$.

Turning this around, if you seek to extend a result on dual maps from Part A Linear Algebra, expect a closure sign to come into play in the normed space setting whenever the proof of the corresponding linear algebra result needs a dimension argument.

8.4. **Example (annihilators and dual maps).**

Define $T \colon \ell^1 \to \ell^1$ by

$$T(x_j) = (\alpha_j) \quad \text{where } \alpha_j = \begin{cases} 2^{-j} x_j & \text{if } j \text{ is odd,} \\ 0 & \text{if } j \text{ is even.} \end{cases}$$

It is routine to check that $T \in \mathcal{B}(\ell^1, \ell^1)$. Since $(\ell^1)^* \cong \ell^\infty$ we can regard $T'$ as a bounded linear operator from $\ell^\infty$ to $\ell^\infty$ [we shall suppress the map $J$ setting up the isometric isomorphism from $\ell^\infty$ onto $(\ell^1)^*$].

We want to find $T'$. Let $e_n = (\delta_{nj})$ in $\ell^1$ and take $(x_j) = e_n$. Let $y = (y_j)$ in $\ell^\infty$ and let $T'y = (\beta_j)$. Then, for any $n$,

$$\sum_{j=1}^\infty \beta_j \delta_{nj} = (T'y)(e_n) = y(Te_n) = \sum_{k=1}^\infty y_{2k+1} 2^{-n} \delta_{n(2k+1)}.$$

It follows from this that

$$\beta_j = \begin{cases} y_j / 2^j & \text{if } j \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that

$$\ker T = \{ (x_j) \in \ell^1 \mid x_j = 0 \text{ for } j \text{ odd} \}$$

and that

$$(\ker T)^\circ = \{ (y_j) \in \ell^\infty \mid y_j = 0 \text{ for } j \text{ even} \}.$$

Take $y = (y_j)$ where $y_j = 1$ for $j$ odd and 0 for $j$ even. Then $y = T'z$ for some $z = (z_j)$ would imply that $z_{2k} = 2^{2k}$, so $z$ could not belong to $\ell^\infty$. We conclude that $y \in (\ker T)^\circ \setminus T'\ell^\infty$. This shows that the inclusion of $T'Y^*$ in $(\ker T)^\circ$ may be strict. Here $T'Y^*$ is not closed.

We now move on to spectral theory. For reasons which will emerge, the usual setting for this will be a complex normed space $X$, but we note that some more elementary results and some examples work just as well when $\mathbb{F} = \mathbb{R}$. At crucial points, clearly flagged, we shall need to assume that $X$ is a Banach space. Thus the best results overall are available in the setting of complex Banach spaces.

8.5. **Definition: the spectrum of a bounded linear operator.**

Let $X$ be a normed space and let $T \in \mathcal{B}(X)$. The **spectrum** of $T$ is

$$\sigma(T) = \{\, \lambda \in \mathbb{C} \mid (\lambda I - T) \text{ is not invertible in } \mathcal{B}(X)\,\}.$$

We may work with either $(\lambda I - T)$ or with $(T - \lambda I)$. It makes no difference which we use.

In identifying $\sigma(T)$ for a particular $T$ and to obtain general properties of the spectrum, we shall exploit results from Section 3, with $T$ replaced by $\lambda I - T$.

8.6. **A closer look at the spectrum of a bounded linear operator.**

We now record what our checklist for invertibility in 3.14 tells us about *non-invertibility* of an operator $\lambda I - T$, for $T \in \mathcal{B}(X)$, where $X$ is a normed space.

The scalar $\lambda \in \sigma(T)$ if and only if at least one of the following holds:

(A) $\lambda$ is an **eigenvalue** of $T$. That is, there exists $x \neq 0$ in $X$ such that $Tx = \lambda x$. Equivalently, $\lambda I - T$ fails to be injective.

(B) $\lambda$ is an **approximate eigenvalue**, meaning that there exists a sequence $(x_n)$ in $X$ such that

$$\|x_n\| = 1 \text{ and } \|Tx_n - \lambda x_n\| \to 0.$$

(C) $\lambda I - T$ fails to be surjective. This happens in particular if $(\lambda I - T)X$ is not dense.

The set of eigenvalues of $T$ is called the **point spectrum** of $T$ and denoted $\sigma_{\mathrm{p}}(T)$. The set of approximate eigenvalues is denoted by $\sigma_{\mathrm{ap}}(T)$ and we refer to it as the **approximate point spectrum**.

The condition in (B) for an approximate eigenvalue $\lambda$ asserts that $\lambda I - T$ could not have a bounded inverse (assuming it were a bijection); recall $(\star)$ in the invertibility checklist. Specifically, this happens if there does not exist $K > 0$ such that

$$\|(\lambda I - T)x\| \geqslant K\|x\|.$$

This implies (consider failure for $K = 1/n$ for $n = 1, 2, \ldots$) there exists a sequence $(u_n)$ with $\|(\lambda I - T)u_n\| < 1/n\|u_n\|$; necessarily $u_n \neq 0$. Then let $x_n = u_n/\|u_n\|$ to show $\lambda$ is an approximate eigenvalue.

We claim that

$$\sigma_{\mathrm{p}}(T) \subseteq \sigma_{\mathrm{ap}}(T) \subseteq \sigma(T).$$

The first inclusion is clear: if $x$ is an eigenvector of norm 1 (wlog) then we may take $x_n = x$ for all $n$. For the second inclusion assume for contradiction that $\lambda \in \sigma_{\mathrm{ap}}(T) \setminus \sigma(T)$ and that $(x_n)$ is as in (B). Hence $S := (T - \lambda I)^{-1}$ exists in $\mathcal{B}(X)$. Then $x_n = S(T - \lambda I)x_n \to S0 = 0$, contradiction.

The following can be useful pointers to identifying approximate eigenvalues:

(i) In sequence spaces it often happens that a particular $\lambda$ fails to be an eigenvalue because any candidate associated eigenvector $x = (x_j)$ would not belong to the space $X$ because it would not have finite norm. Try 'truncating': consider $(x^{(n)})$, where $x_j^{(n)} = x_j$ for $j \leqslant n$ and 0 otherwise, to show that $\lambda$ is an approximate eigenvalue.

(ii) Given $y \in X$ try to solve the equation $(\lambda I - T)x = y$ for $x$. If this can be done, then consider whether $(\lambda I - T)^{-1}$ is bounded. If it's not you may be able to find a sequence $(x_n)$ which witnesses this.

Now for some first examples.

8.7. **Example 1 (a multiplication operator).**

Let $X$ be the space $C(K)$ of continuous complex-valued functions on a compact subset $K$ of $\mathbb{C}$, with the sup norm. Define $T \in \mathcal{B}(X)$ by

$$(Tf)(z) = zf(z) \quad (z \in K).$$

Then $\sigma(T) = K$. This follows immediately from Example 3.16(1), replacing $T$ by $\lambda I - T$ in that example.

8.8. **Example 2 (eigenvalues and approximate eigenvalues).**

Let $T \in \mathcal{B}(c_0)$ be given by $T(x_j) = (x_j/j)$. Each $\lambda = 1/n$ is an eigenvalue with $e_n$ as an associated eigenvector $(n = 1, 2, \ldots)$.

Certainly 0 is not an eigenvalue because $\ker T = \{0\}$. However 0 is an approximate eigenvalue: consider $x_n = e_n$ and note $\|Te_n\| = 1/n \to 0$.

Consider $\lambda \notin S := \{0\} \cup \{1/n \mid n \geqslant 1\}$. Then $S$ is closed and bounded, hence there exists $\delta > 0$ such that $|\lambda - s| \geqslant \delta$ for all $s \in S$. Let $y = (y_j) \in c_0$ and suppose $x = (x_j)$ is such that $(\lambda I - T)x = y$. This requires $(\lambda - 1/j)x_j = y_j$ for all $j$. This gives $x_j$ such that $|x_j| \leqslant |y_j|/\delta$. From this, $x \in c_0$ and $\|\lambda I - T)^{-1}\| \leqslant \delta^{-1}$.

We conclude that $\sigma(T) = S$.

We now work towards establishing general properties of the spectrum. Here completeness of $X$ becomes important. The following elementary lemma will be useful when we need to juggle with invertible operators.

8.9. **A lemma on invertible operators.**

(i) *Let $X$ be a normed space and $P, Q \in \mathcal{B}(X)$.*
  (a) *Assume $P$ and $Q$ are invertible. Then $PQ$ is invertible with inverse $Q^{-1}P^{-1}$.*
  (b) *Assume that $PQ = QP$, $P$ is invertible and that $PQ$ is invertible. Then $Q$ is invertible.*

(ii) *Let $X$ be a Banach space and let $P, Q \in \mathcal{B}(X)$. Assume that $P$ is invertible and that*

$$\|P - Q\| < \|P^{-1}\|^{-1}.$$

*Then $Q$ is invertible.*

*Proof.* We leave (i)(a) is an easy exercise; remember that for an operator to be invertible we need a 2-sided inverse, in $\mathcal{B}(X)$.

Consider (i)(b). Assume $S$ is an inverse for $PQ$ in $\mathcal{B}(X)$. Then $I = (PQ)S = S(PQ)$. Since multiplication is associative and $P$ and $Q$ commute, $I = Q(PS) = (SP)Q$. We deduce that $Q$ is a bijection. Invertibility of $PQ$ implies that there exists $\delta > 0$ such that $\|PQx\| \geqslant \delta\|x\|$ for all $x$. Then

$$\delta\|x\| \leqslant \|PQx\| \leqslant \| \leqslant \|P\|\|Qx\|.$$

Also $PQ$ invertible forces $P \neq 0$, so $\|P\| \neq 0$. Therefore $\|Qx\| \geqslant (\delta/\|P\|)\|x\|$ for all $\|x\|$. We now deduce from 3.14 that $Q$ in invertible.

[Note that, purely algebraically, we got a left inverse and a right inverse for $Q$, each of which is a bounded operator. *But we don't know that these are equal.* Hence we need to argue via the invertibility checklist.]

We now prove (ii). Consider $R = (P - Q)P^{-1}$. Then $R \in \mathcal{B}(X)$ and $\|R\| \leqslant \|P - Q\|\|P^{-1}\|$ and so $\|R\| < 1$. Hence by Proposition 3.17, $I - R$ is invertible. Then $(I - R)P$ is invertible too. But $(I - R)P = Q$. $\square$

We already have enough information to prove quite a lot about the spectrum of a bounded operator on a Banach space. A corresponding result holds when the scalar field is $\mathbb{R}$.

8.10. **Theorem I (basic facts about spectrum).** *Let $T$ be a bounded linear operator on a complex Banach space $X$. Then*

(i) *$\sigma(T) \subseteq \overline{\mathrm{D}}(0, \|T\|)$, the closed disc center $0$ radius $\|T\|$.*

(ii) *$\sigma(T)$ is closed.*

(iii) *$\sigma(T)$ is a compact subset of $\mathbb{C}$.*

*Proof.* Proposition 3.17, applied to $I - T/\lambda$ ($\lambda \neq 0$), implies that $\lambda I - T$ is invertible if $|\lambda| > \|T\|$. Hence (i) holds.

Now assume $\lambda \notin \sigma(T)$. Then

$$\|(\lambda I - T) - (\mu I - T)\| = |\lambda - \mu| < \|(\lambda I - T)^{-1}\|^{-1}$$

if $\mu$ is such that $|\lambda - \mu|$ is sufficiently small. Now use Lemma 8.9(ii).

Part (ii) tells us that $\mathbb{C} \setminus \sigma(T)$ is an open set, so $\sigma(T)$ is closed. Since it is also bounded, by (i), the Heine–Borel Theorem gives (iii). $\square$

The next example illustrates how the results in Theorem I can be combined to identify the spectrum of an operator in certain cases.

8.11. **Example 3 (left-shift operator on $\ell^1$).**

Let $T\colon \ell^1 \to \ell^1$ be given by

$$T(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots).$$

Then (see 3.3(1)) $\|T\| = 1$, so $\sigma(T) \subseteq \overline{\mathrm{D}}(0, 1)$.

Now find eigenvalues (if any). Suppose $x = (x_j) \in \ell^1$ is such that $Tx = \lambda x$. Equating components:

$$\lambda x_j = x_{j+1} \quad (j \geqslant 1).$$

Hence $x_j = \lambda^{j-1}x_1$. If $x_1 = 0$, we get $x = 0$ and this is not an eigenvector. Otherwise $x = (x_j) \in \ell^1$ iff $|\lambda| < 1$. Hence $\sigma_{\mathrm{p}}(T) = \mathrm{D}(0, 1)$.

Finally

$$\mathrm{D}(0, 1) = \sigma_{\mathrm{p}}(T) \subseteq \sigma(T) \subseteq \overline{\mathrm{D}}(0, 1).$$

Taking the closure right through and using the fact that $\sigma(T)$ is closed we get

$$\sigma(T) = \overline{\mathrm{D}}(0, 1).$$

The next result can give valuable information about $\sigma(T)$ in cases where $\|T^n\|$ can be found explicitly and where Theorem I together with knowledge of $\sigma_p(T)$ does not pin down $\sigma(T)$ completely; contrast Example 4 below with Example 3.

8.12. **Proposition (a refinement of Theorem I(ii)).** *Let $T \in \mathcal{B}(X)$, where $X$ is Banach. Then*

$$\lambda \in \sigma(T) \implies |\lambda|^n \leqslant \|T^n\| \text{ for all } n.$$

*Proof.* Take $n \geqslant 1$. By 8.10(i) applied to $T^n$ it will be enough to prove that

$$\lambda \in \sigma(T) \implies \lambda^n \in \sigma(T^n).$$

We shall obtain the contrapositive. Assume $\lambda^n I - T^n$ is invertible with inverse $S$. We claim $\lambda I - T$ is invertible. Observe that

$$\lambda^n I - T^n = (\lambda I - T)p(T) = p(T)(\lambda I - T),$$

where $p(T)$ is a polynomial in $T$ and belongs to $\mathcal{B}(X)$. Now apply 8.9(i)(b). $\square$

8.13. **Example 4 (an integral operator).** [From FHS Paper B4a:3]

Let $X = C[0, 2\pi]$ (real-valued functions), with the supremum norm. Define $T$ by

$$(Tx)(t) = \int_0^t \sin s\, x(s)\, \mathrm{d}s.$$

Then $T$ is a bounded linear operator on $X$ (recall 3.3(4)) and an easy induction gives, for $n \geqslant 1$,

$$|(T^n x)(t)| \leqslant \frac{t^n}{n!}\|x\|_\infty.$$

Hence, taking the sup first over $t$ and then over $x$, we get

$$\|T^n\| \leqslant \frac{(2\pi)^n}{n!}$$

The RHS $\to 0$ as $n \to \infty$ (why?). Hence by Proposition 8.12 the only possible value for $\lambda$ when $\lambda \in \sigma(T)$ is $\lambda = 0$.

We assert that 0 is an approximate eigenvalue. This is witnessed by the sequence $(x_n)$ where $x_n(t) = \cos nt$. To confirm this, note that $\|x_n\| = 1$ and integrate by parts to get an upper bound for $\|Tx_n\|$ from which we can deduce $\|Tx_n - 0 \cdot x_n\| = \|Tx_n\| \to 0$. We conclude that $\sigma(T) = \{0\}$.

We have already seen that looking at the powers of an operator $T$ may provide valuable information about its spectrum. We now take this idea further. Here we don't need $X$ to be a Banach space but we do need the scalar field to be $\mathbb{C}$.

8.14. **Theorem II (Spectral Mapping Theorem for polynomials).** *Let $p$ be a complex polynomial, not identically zero. Let $X$ be a complex normed space and $T \in \mathcal{B}(X)$. Then*

$$\sigma(p(T)) = p(\sigma(T)) := \{\, p(\lambda) \mid \lambda \in \sigma(T)\,\}.$$

*Proof.* There is nothing to prove if $p$ is a constant so assume $p$ has degree $n \geqslant 1$. Let $\mu \in \mathbb{C}$. We can factorise $p(z) - \mu$ as a product of linear factors:

$$p(z) - \mu = \alpha(z - \beta_1)\cdots(z - \beta_n),$$

for some $\alpha \neq 0$ and $\beta_1, \ldots, \beta_n$. Then

$$p(T) - \mu I = \alpha(T - \beta_1 I)\cdots(T - \beta_n I).$$

Here the factors commute and $\mu = p(\lambda)$ for some $\lambda$ if and only if $\lambda \in \{\beta_1, \dots, \beta_n\}$. So

$$\begin{aligned} \mu \in p(\sigma(T)) &\iff \exists \lambda \in \sigma(T) \text{ such that } \mu = p(\lambda) \\ &\iff \sigma(T) \cap \{\beta_1, \dots . \beta_n\} \neq \emptyset. \end{aligned}$$

Assume $\mu \notin \sigma(p(T))$. Then $\lambda \neq \beta_r$ and hence $(T - \beta_r I)$ is invertible, for each $r$. This implies $\alpha(T - \beta_1 I) \cdots (T - \beta_n I)$ is invertible. We deduce that $\mu \notin p(\sigma(T))$.

Assume $\mu \notin p(\sigma(T))$. Then, for each $r$,

$$p(T) - \mu I = (T - \beta_r I) \prod_{j \neq r} (T - \beta_j I).$$

It follows from Lemma 8.9(i)(b) that $T - \beta_r I$ is invertible. Therefore $\beta_r \notin \sigma(T)$ for $r = 1, \dots, n$. Hence $\mu \notin p(\sigma(T))$. □

8.15. **Example 5 (use of SMT).**

Let $T \in \mathcal{B}(\ell^1)$ be given by

$$T(x_j) = (x_j - 2x_{j+1} + x_{j+2}).$$

Then $T = (I - L)^2$, where $L$ is the left-shift operator considered in Example 3: $\sigma(L) = \overline{\mathbb{D}}(0,1)$. By Spectral Mapping Theorem,

$$\sigma(T) = \{\, (1 - \lambda)^2 \mid |\lambda| \leqslant 1 \,\}.$$

Describing $\sigma(T)$ explicitly is now an exercise on the geometry of the complex plane. By going into polar coordinates, one can show that

$$\sigma(T) = \{\, (r, \theta) \mid 0 \leqslant \theta < 2\pi,\ 0 \leqslant r \leqslant 2 + 2\cos\theta \,\}.$$

The boundary curve is a cardioid. Moral: quite simple polynomial transformations can change the spectrum in ways which would have been hard to predict.

Our earlier Proposition 8.12 is a corollary of the Spectral Mapping Theorem. A much stronger result linking the spectrum of $T$ to powers of $T$ can be proved than that in Proposition 8.12. We omit the proof of the **Spectral Radius Formula**, which needs more advanced theory than that in FA-I and which we shall not need.

8.16. **Theorem (Spectral Radius Formula).** *Let $X$ be a complex Banach space. Then the spectral radius of $T$, defined by*

$$\operatorname{rad}(\sigma(T)) = \sup\{\, |\lambda| \mid \lambda \in \sigma(T) \,\},$$

*is given by*

$$\operatorname{rad}(\sigma(T)) = \inf_n \|T^n\|^{1/n} = \lim_{n \to \infty} \|T^n\|^{1/n}.$$

We have flagged up already that it can be useful to consider the dual map $T'$ when seeking information about a bounded linear operator $T \in \mathcal{B}(X)$—assuming of course we can (i) describe the dual space $X^*$ on which $T'$ acts and (ii) describe the action of $T'$ explicitly. But even with these provisos concerning its usefulness in specific cases the following theorem is of interest and the proof instructive. It is crucial that we work with a Banach space since we shall call on Proposition 3.15.

8.17. **Theorem II ($T$ and $T'$ in tandem).** *Let $X$ be a Banach space. Let $T \in \mathcal{B}(X)$ and let $T' \in \mathcal{B}(X^*)$ be the dual operator. Then*

$$\sigma(T) = \sigma_{\mathrm{ap}}(T) \cup \sigma_p(T').$$

*Proof.* The proof depends on Proposition 3.15 and results from 8.3 as they apply to $P := \lambda I - T$. Note that $P' = \lambda I - T'$. Also we shall use the fact (from Problem sheet Q. 23) that $\sigma(T') = \sigma(T)$.

Hence, from 8.6,

$$\sigma(T) \supseteq \sigma_{\mathrm{ap}}(T) \cup \sigma_{\mathrm{p}}(T').$$

For the proof of the reverse inclusion, assume $\lambda \notin \sigma_{\mathrm{ap}}(T) \cup \sigma_{\mathrm{p}}(T')$. Then there exists $K > 0$ such that $\|Px\| \geqslant K\|x\|$ for all $x$. Also $\ker P' = \{0\}$ so, from 8.3, $PX$ is dense. Now Proposition 3.15 implies that $P$ is invertible. $\square$

Something's missing! So far in all our examples we have seen that the spectrum is non-empty. But is this true in general? Our final piece of theory will show that if $X$ is a complex Banach space, then $\sigma(T) \neq \emptyset$ for any $T \in \mathcal{B}(X)$. We shall treat the proof of this quite lightly, aiming to give the flavour without the fine detail. First we need some preliminaries.

8.18. **The resolvent set.**

Let $T \in \mathcal{B}(X)$, where $X$ is a complex Banach space. Then the **resolvent set** is

$$\rho(T) := \mathbb{C} \setminus \sigma(T).$$

For $\lambda \in \rho(T)$ we can define a bounded linear operator $R(\lambda, T) = (T - \lambda I)^{-1}$. For $|\lambda| > \|T\|$ this is given by

$$R(\lambda, T) = \sum_{k=0}^{\infty} \lambda^{-k-1} T^k;$$

this comes from Proposition 3.17 by rescaling.

We also have, for $\lambda, \mu \in \rho(T)$,

$$(T - \mu I) = (T - \lambda I) + (\lambda - \mu)I = (T - \lambda I)\left(I + (\lambda - \mu)(T - \lambda I)^{-1}\right).$$

We can deduce from this that the map $\lambda \mapsto (T - \lambda I)^{-1}$ is a continuous function on $\rho(T)$.

Moreover we have the **Resolvent Identity**

$$R(\lambda, T) - R(\mu, T) = (\lambda - \mu) R(\lambda, T) R(\mu, T)$$

(proof is pure linear algebra).

8.19. **Theorem IV (non-empty spectrum).** *Let $X$ be a complex Banach space and $T \in \mathcal{B}(X)$. Then $\sigma(T) \neq \emptyset$.*

*Proof.* We argue by contradiction. If the conclusion is false $R(\lambda, T) = (\lambda I - T)^{-1}$ is a bounded linear operator for every $\lambda \in \mathbb{C}$. By the Hahn–Banach Theorem there exists $f \in (\mathcal{B}(X))^*$ such that $f(R(0, T)) \neq 0$. Consider the map

$$\varphi \colon \lambda \mapsto f(R(\lambda, T)) \quad (\lambda \in \mathbb{C}).$$

This is a composition of continuous maps, so continuous. But, for $\lambda \neq 0$,

$$R(\lambda, T) = (-\lambda(I - \lambda^{-1}T))^{-1} = -\lambda^{-1}(I - \lambda^{-1}T)^{-1} \to 0 \quad \text{as } |\lambda| \to \infty.$$

To confirm this, note that $(I - \lambda^{-1}T)^{-1} \to I$.

Provided we can prove that $\varphi$ is holomorphic, then we can conclude from Liouville's Theorem that $\varphi \equiv 0$ and this would contradict the fact that $\varphi(0) \neq 0$. With some juggling we can show with the aid of the Resolvent Identity that $\lambda \mapsto \varphi(\lambda)$ has a convergent power series expansion in a suitably small neighbourhood of each $\mu \in \mathbb{C}$. Hence it is holomorphic. □

8.20. **SUMMARY: Pinning down $\sigma(T)$.**

The second column in the following table gives a reference for the may assist in determining $\sigma(T)$. Which of the various techniques are needed in a particular example will depend on the type of operator under consideration and what information one finds along the way.

TABLE 1. Strategies for finding spectrum

| | | Theory that is relevant or maybe useful |
|---|---|---|
| | Methods not involving (B), (C) in 8.6 | |
| 1. | Calculate $\|T\|$ | Theorem I(i) |
| 2. | Find $\sigma_{\mathrm{p}}(T)$ & its closure | Theorem I |
| 3. | If expedient, investigate $\|T^n\|$: | 8.12 |
| | if $\inf_n \|T^n\|^{1/n} = 0$, then $\sigma(T) = \{0\}$ | 8.12 & Theorem IV |
| 4. | Is $T$ a polynomial of a simpler operator? | SMT (Theorem II) |
| | More refined analysis using (B), (C) | |
| | Assume $\lambda \notin \sigma_p(T)$ | |
| 5. | Try to solve $y = (\lambda I - T)x$ for $x$: | |
| | $\exists y$ s.t. $\nexists x \in X$? | If so, surjectivity fails & $\lambda \in \sigma(T)$ |
| 6. | Is $\lambda \in \sigma_{ap}(T)$? | If so, $\lambda I - T$ cannot have a bounded inverse & $\lambda \in \sigma(T)$ |
| 7. | Consider $T'$ | Theorem III |

Overall, we seek to find for which values of $\lambda$ the operator $(\lambda I - T)$

- does have a bounded inverse;
- fails to have a bounded inverse.

Our strategy in practice may be to toggle between looking for when $\lambda \notin \sigma(T)$ and when $\lambda \in \sigma(T)$ until all $\lambda \in \mathbb{C}$ are identified as being in or out of the spectrum.

Towards finding when a bounded inverse does exist for $\lambda I - T$, we may first seek a disc to which $\sigma(T)$ must be confined [1. and 3. in the table].

In the other direction, to find points $\lambda$ which have to belong to $\sigma(T)$, we first look for the eigenvalues. As usual we try to find conditions on $\lambda$ so that $Tx = \lambda x$ has a non-zero solution for $x$. Crucially we require $x$ *in* $X$. When this membership test fails for a candidate value of $\lambda$ we may suspect that $\lambda \in \sigma_{ap}(T)$. In some cases $\sigma_p(T)$ may be dense in $\mathrm{D}(0, \|T\|)$, giving $\sigma(T) = \mathrm{D}(0, \|T\|)$ [as in Example 3]. Sometimes this won't hold, but a given operator may be a polynomial in some other operator for which the spectrum can be easily found. Then the Spectral Mapping Theorem comes into play [4. in the table; Example 5].

In certain cases there may be few eigenvalues or none at all [the right-shift operator on $\ell^1$ provides an example [see Problem sheet Q. 26]. For $\lambda \notin \overline{\sigma_p(T)}$ and where $\lambda I - T$ is not already guaranteed by 1. or 3. to be invertible, it may then be worth trying to find an inverse map $(\lambda I - T)^{-1}$: could it have domain $X$, i.e., is $\lambda I - T$ surjective? [5. in the table]. If surjectivity is assured, one then needs to test for boundedness of the inverse (($\star$) in 3.14). [Illustrations: Examples 1 and 2.]

To show $\lambda \notin \overline{\sigma_p(T)}$ is such that $\lambda \in \sigma(T)$ we want to find whether one of (B), (C) in 8.6 holds. There can be points in the spectrum for which $\lambda$ is not an eigenvalue or even an approximate eigenvalue of $T$ and failure of surjectivity may be hard to prove directly. If $X$ is a space whose dual space is known (for example $X = \ell^p$ with $1 \leqslant p < \infty$ (recall 7.3)), then recourse to Theorem III, involving $T'$, may be a good option [7. in the table]: it may be easier to find eigenvalues of $T'$ than to explore when $\lambda I - T$ is surjective. [See Problem sheet Q. 26 and also the bonus extension question on the final problem sheet.]

8.21. **What use is the spectrum?**

That's a story for another, more advanced course. What we'd hope to do is to extend to suitable operators on infinite-dimensional complex Banach spaces spectral representations such as we have for self-adjoint operators on finite-dimensional inner product spaces. We'd expect, demanding on how complicated the spectrum is, to need a spectral representation involving an infinite sum or even a Stieltjes-style integral.

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