Professor Joyce **B3.2 Geometry of Surfaces** **MT 2019**

Initial problem sheet. Solutions

1. Define an equivalence relation on the unit interval $[0,1]$ by $x \sim y$ if either $x = y$ or $x = 0$ and $y = 1$. Show that the set of equivalence classes with the quotient topology is homeomorphic to the unit circle.

Solution. Map $[0,1]/\sim \to S^1$ by $x \mapsto (\cos 2\pi x, \sin 2\pi x)$. This maps $0, 1$ to $(1,0)$, and is otherwise injective. It is clearly a homeomorphism.

2. Let X be the space of equivalence classes of points in $\mathbb{R}^2 \backslash \{0\}$ under the equivalence relation $(x_1, x_2) \sim (\lambda x_1, \lambda^{-1} x_2)$ for some $\lambda \in \mathbb{R}\backslash\{0\}$. Show by considering the equivalence classes of $(0,1)$ and $(1,0)$ that the space X is not Hausdorff in the quotient topology.

Solution. Write $[x,y]$ for the equivalence class of (x,y). Then $[1,0] \neq [0,1]$. Let U, V be open neighbourhoods of $[1,0]$ and $[0,1]$ in X. Then there exist $\delta, \epsilon \in (0,1)$ such that $[x,y] \in U$ for $(x,y) \in B_\delta(1,0)$ and $[x,y] \in U$ for $(x,y) \in B_\delta(0,1)$. Thus if $0 < \gamma < \min(\delta, \epsilon)$ then $[1,\gamma] \in U$ and $[\gamma, 1] \in V$. But taking $\lambda = \gamma^{-1}$ shows that $[1,\gamma] = [\gamma, 1]$, so $U \cap V = \emptyset$. Thus X is not Hausdorff.

Remark. X does have the property that every $x \in X$ has an open neighbourhood homeomorphic to an open set in $\mathbb{R}$. So not being Hausdorff is the only thing preventing X being a topological manifold.

Can you draw a picture of X?

3. A connected surface X is obtained by taking n copies of a sphere with two disjoint open discs removed, and identifying the $2n$ boundary circles in pairs. Show that the Euler characteristic of X must vanish.

Solution. The sphere with two disjoint open discs removed is an annulus $[0,1] \times S^1$. There is an obvious division into $V = 2$ vertices, and $E = 3$ edges, and $F = 1$ faces, such that each boundary circle has 1 vertex and 1 edge.

When n copies of the annulus are glued together, the result has V' faces, E' edges, and F' faces, where

$$V' = nV - n = n, \quad E' = nE - n = 2n, \quad F' = nF = n,$$

and we subtract $-n$ from V', E' to avoid double-counting the edges and vertices on the boundry circles. Hence X has Euler characteristic $\chi(X) = V' - E' + F' = n - 2n + n = 0$.

4. A connected surface Y is obtained by taking $2n$ copies of a sphere with *three* disjoint open discs removed, and identifying the $6n$ boundary circles in pairs. What values can the Euler characteristic take?

Can you make this surface this way?

Solution. Follow the method of question 3. The sphere with three disjoint open discs removed has a division into $V = 4$ vertices, and $E = 6$ edges, and $F = 1$ faces, such that each boundary circle has 1 vertex and 1 edge. When $2n$ copies of this are glued together, the result has V' faces, E' edges, and F' faces, where

$$V' = 2nV - 3n = 5n, \quad E' = 2nE - 3n = 9n, \quad F' = 2nF = 2n,$$

so $\chi(X) = V' - E' + F' = 5n - 9n + 2n = -2n$.

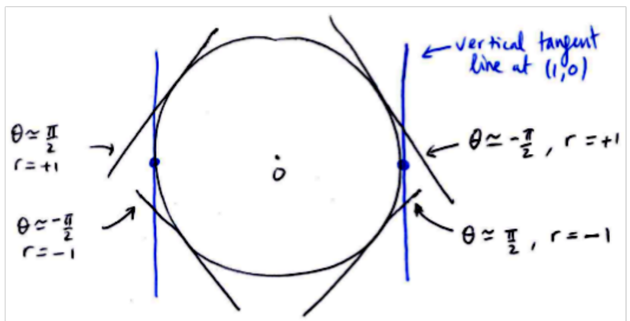
If X is connected and oriented it has g holes with $2-2g = -2n$, so $g = n+1$. When $n = 1$ this gives a surface with genus 2 as in the picture. We can cut it into two spheres with 3 holes by a plane through the centre of both holes.

5. Let S be the set of all straight lines in $\mathbb{R}^2$ (not necessarily through 0). Show that there is a natural way to make S into a topological surface. Show that S is homeomorphic to the open Möbius band M.

Solution 1.
Lines in $\mathbb{R}^2$ are given by an equation $ax + by = c$. We only care about $(a, b, c) \in \mathbb{R}^3$ up to rescaling by $\mathbb{R} \setminus 0$, so that is a point $[a : b : c] \in \mathbb{RP}^2$. But we do not want a, b to vanish simultaneously, so we get $\mathbb{RP}^2$ minus the point $[0 : 0 : 1]$. But $\mathbb{RP}^2$ minus a point is homeomorphic to $\mathbb{RP}^2$ minus a disc, which we saw (in Exercise 2) is an open Möbius band.

Solution 2.
Given a line L, nearby lines differ from L by a tilting angle and by varying the distance of the line from the origin. So we declare that the angle and the (appropriately signed) distance define continuous local coordinates.

Indeed, for the subset of all non-vertical lines we can define a tilting angle $\theta \in (-\pi/2, \pi/2)$ with respect to the positive x-axis and a signed distance $r \in \mathbb{R}$ from the origin (with signs as explained in the footnote), and for the subset of all non-horizontal lines we instead use the tilting angle ψ with respect to the negative y-axis (turn the picture by 90 degrees). One could work with these two patches and compare the coordinates (so $\psi = \pi - \theta$ and the signed distance may or may not flip sign). But it's easier to think of the vertical lines as left- and right-limits of the coordinates (θ, r).



Indeed, consider all lines at absolute distance 1 from the origin: so all the lines tangent to the unit circle. The tangent at $(1, 0)$ is vertical, the lines tangent just above at $(1 - \text{positive}, \text{positive}) \in S^1$ have $(\theta, r) \simeq (-\pi/2, +1)$, whereas the lines tangent just below at $(1 - \text{positive}, \text{negative}) \in S^1$ have $(\theta, r) \simeq (\pi/2, -1)$. So we should identify

$$(-\pi/2, +1) \sim (\pi/2, -1)$$

with the vertical tangent line at $(1, 0)$. Similarly, at $(-1, 0)$ we deduce that we should identify

$$(\pi/2, +1) \sim (-\pi/2, -1)$$

with the vertical tangent line at $(-1, 0)$. The fact that we used radius $|r| = 1$ is irrelevant, the same argument works for any positive radius. Finally, for $|r| = 0$, we have the space of lines through 0, described by the tilting angles $[-\pi/2, \pi/2]$ where $-\pi/2$ and $\pi/2$ both correspond to the vertical y-axis. So we identify $(-\pi/2, 0) \sim (\pi/2, 0)$, consistently with the previous identifications.

In conclusion:

$$S = [-\pi/2, \pi/2] \times \mathbb{R}/ \left((\pi/2, r) \sim (-\pi/2, -r) \text{ for all } r \in \mathbb{R}\right).$$

Using the homeomorphism $\mathbb{R} \to (-\pi/2, \pi/2)$, $x \mapsto \tan^{-1}(x)$, we see that S is the quotient of the square $[-\pi/2, \pi/2] \times (-\pi/2, \pi/2)$ by the same relation as above. Adding $(\pi/2, \pi/2)$ and rescaling by $1/\pi$ shows that S is homeomorphic to M.

6. Consider the quotient

$$S = \mathbb{R}^2/G$$

where $G = \mathbb{Z}^2$ acts[1] by $(n, m) \bullet (x, y) = ((-1)^m x + n, y + m)$ on $\mathbb{R}^2$, where $n, m \in \mathbb{Z}$. Show that S is homeomorphic to the Klein bottle.

Solution. Locally, near $p \in S$ we pick a representative $(x, y) \in \mathbb{R}^2$ so $p = [x, y]$. Then a small enough open disc D_p around p (for example, of radius 1/100) will have no two points lying in the same G-orbit. Thus the quotient map $f : \mathbb{R}^2 \supset D_p \mapsto \{[q] : q \in D_p\} \subset S$ is bijective, and by definition f, f^{-1} are continuous, so f is a homeomorphism. So S is a topological surface, and we declare that the two coordinates of $\mathbb{R}^2$ near (x, y) are local smooth coordinates near p. Two different choices of representative (x, y), $((-1)^m x + n, y + m)$ yield local coordinates which differ by a smooth map, namely $\mathbb{R}^2 \to \mathbb{R}^2$, $(X, Y) \mapsto ((-1)^m X + n, Y + m)$. So S is a smooth surface.

Notice that there is a representative of each point of the orbit space $\mathbb{R}^2/G$ inside the unit square $[0, 1] \times [0, 1]$. In the interior, no two points lie in the same G-orbit, whereas on the boundaries the identifications coming from the G-action prescribe precisely the gluing of edges of a square used in the lecture notes to define the Klein bottle as a topological surface. So $\mathbb{R}^2/G$ can be homeomorphically identified with K, by mapping each G-orbit in $\mathbb{R}^2$ to a representative of that G-orbit lying in the unit square.

7. A figure 8 loop consists of two circles touching at a point. Show that a torus can be obtained by attaching a disc onto a figure 8 loop.

Solution. Recall from the notes that the torus arises from the square by identifying parallel edges. The two pairs of parallel edges form two loops, which touch at a point (corresponding to the four identified vertices of the square). Let's call those loops A, B (oriented by, say, going clockwise around the boundary of the square). The closed square is homeomorphic to a closed disc, so T^2 is obtained by attaching the closed disc $\mathbb{D}^2$ by going around the loops A, B, A^{-1}, B^{-1}, where A^{-1} means you go around the loop A in the opposite direction.

[1] $G = \mathbb{Z}^2$ as a set, but as a group $G = \mathbb{Z} \rtimes \mathbb{Z}$ is a *semi-direct product*.