Professor Joyce **B3.2 Geometry of Surfaces** **MT 2019**

Problem Sheet 1

1. Define the Euler characteristic of a surface with a subdivision. By choosing a suitable subdivision show that the Euler characteristic of a torus is zero.

An engineer constructs a vessel in the shape of a torus from a finite number of steel plates. Each plate is in the form of a not necessarily regular curvilinear polygon with n edges. The plates are welded together along the edges so that at each vertex n distinct plates are joined together, and no plate is welded to itself. What is the number n? Justify your answer.

[*You may assume that the Euler characteristic is independent of the choice of subdivision of the surface.*]

2. (*The Thomsen graph or the Three Amenities Problem*)

Let H_1, H_2, H_3, G, W, E be six points on a sphere. Show that it is not possible to join each of H_1, H_2, H_3 to each of G, W, E by curves intersecting only at their end points (nine curves in all).

[*You may assume that such a configuration of curves would give a subdivision of the sphere.*]

By drawing a diagram show that such a construction is possible on the projective plane. Decide whether it is possible on the torus or the Klein bottle.

3. Use the formula for the Euler characteristic to show that there are no more than five Platonic solids.

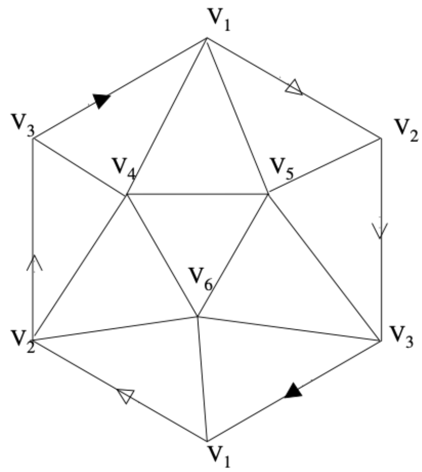
(A Platonic solid is a convex polyhedron with congruent faces consisting of regular polygons and the same number of faces meet at each vertex.)

What are the possibilities for subdividing a torus into polygons each with n sides, and such that k edges meet at each vertex?

4(i) Calculate the Euler characteristic of the surface given in planar form by $a_1a_2a_3^{-1}a_4a_5a_2^{-1}a_1^{-1}a_4^{-1}a_3a_5$. Show that the surface contains a Möbius band.

(ii) By looking for $xyx^{-1}y^{-1}$ terms (or using the classification of surfaces) show that the surface described by $b_1a_2b_3a_3^{-1}b_3^{-1}a_3a_2^{-1}a_1^{-1}b_1^{-1}a_1$ is homeomorphic to $T\#T$.

5. Take a hexagon and identify the opposite edges to form the projective plane P.



(i) For the triangulation of P described in the picture, show that there is a unique edge joining any two of the six vertices $V_1, \ldots, V_6$ in $\mathbb{R}^2$.

(ii) Now let $e_1, e_2, \ldots, e_6$ be the standard basis vectors of $\mathbb{R}^6$ and define a continuous map $f : P \to \mathbb{R}^6$ by

- $f(V_i) = e_i$ on the vertices
- $f(sV_i + tV_j + uV_k) = se_i + te_j + ue_k$ (where $0 \leq s, t, u \leq 1$ and $s+t+u = 1$) on the face with vertices $\{V_i, V_j, V_k\}$

Show that this map is a homeomorphism onto its image. Show further that any hyperplane $\sum_1^6 a_i x_i = b$ in $\mathbb{R}^6$ divides $f(P)$ into at most two connected components.