**Professor Joyce** **B3.2 Geometry of Surfaces** **MT 2019**

**Problem Sheet 2**

**1.** Let $f : X \to Y$ be a holomorphic map of compact connected Riemann surfaces of degree 1.

**(i)** Show that $f$ has no ramification points.

**(ii)** Show that $f$ is a homeomorphism.

**(iii)** Show that $f^{-1}$ is holomorphic.

**2.** Let $f : X \to Y$ be a nonconstant holomorphic map of compact connected Riemann surfaces, where $X$ is the Riemann sphere. Use the general form of the Riemann-Hurwitz formula to deduce that $Y$ is homeomorphic to $X$.

**3.** The *Korteweg–de Vries equation* which describes shallow water waves is

$$\frac{\partial \phi}{\partial t} + \frac{\partial^3 \phi}{\partial x^3} + 6\phi \frac{\partial \phi}{\partial x} = 0.$$

**(i)** A solution with a fixed wave form is given by $\phi(x,t) = f(x - ct)$. Show that $f$ satisfies the equation

$$-cf' + f''' + 6ff' = 0.$$

**(ii)** Using the relation $(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$ find constants $a, b$ such that $f = a\wp + b$ satisfies this equation where $\wp$ is the Weierstrass $\wp$-function. Can you describe the sort of wave this corresponds to?

**4.** Let $f : X \to Y$ be a holomorphic map of compact connected Riemann surfaces of degree 2. Show that there is a non-trivial holomorphic homeomorphism $\sigma : X \to X$ such that $f \circ \sigma = f$ and $\sigma^2$ is the identity map. How many fixed points does your map have?