B2.1 Introduction to Representation Theory
Problem Sheet 2, MT 2019

1. Let A be the three-dimensional $\mathbb{R}$-algebra of all upper triangular 2×2 matrices over $\mathbb{R}$. Find a composition series of the A-module A. Show that A has two simple A-modules (up to isomorphism), and that one of them occurs twice as a composition factor in your composition series.

2. The **radical** $\mathrm{rad}V$ of an A-module V is defined to be the intersection of all maximal submodules of V. Let A be an algebra and consider the A-modules and A-submodules $V \subseteq M_1, M_2 \subseteq X$.

   (a) Show that $M_1/V \cap M_2/V = (M_1 \cap M_2)/V$.

   (b) Suppose that V is finite dimensional. Show that $V/\mathrm{rad}(V)$ is semi-simple.

   (c) Show that $\mathrm{rad}(V)$ is the smallest submodule W of V with V/W semi-simple.

3. Let G be a finite group and N a normal subgroup of G. Let V be a simple KG-module. View V as KN-module by restriction of the action. Prove that V as KN-module is semi-simple.

4. Suppose V is an A-module with two composition series, say $0 \subset U \subset V$ and $0 \subset W \subset V$ and where $U \neq W$.

   (a) Show that $V = U \oplus W$ as A-modules.

   (b) Now assume that U and W are isomorphic, let $\psi : U \to W$ be an A-module isomorphism. For $\lambda \in K$ fixed, define

   $$U_\lambda := \{u + \lambda\psi(u) \mid u \in U\}.$$

   Check that U_λ is a submodule of V and that it is isomorphic to U.

   (c) Deduce that V has infinitely many composition series when K is infinite.

5. Let $A = \mathbb{C}G$ be the group algebra of the dihedral group of order 10,

   $$G = D_{10} = \langle \sigma, \tau : \sigma^5 = 1,\ \tau^2 = 1, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle.$$

   Suppose ζ is a 5-th root of 1 (and $\zeta \neq 1$). You may assume that the matrices

   $$\rho(\sigma) = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, \quad \rho(\tau) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

   satisfy the defining relations for G, hence give a group homomorphism $\rho : G \to GL_2(\mathbb{C})$.

   (a) Prove that the representation ρ is irreducible, (that is the A-module V corresponding to the representation ρ is simple).

(b) Suppose G is any finite group, and $\rho_1,\ \rho_2 : G \to GL_n(\mathbb{C})$ are representations. Show that if ρ_1, ρ_2 are equivalent then for all $g \in G$, we have $\mathrm{tr}(\rho_1(g)) = \mathrm{tr}(\rho_2(g))$ where $\mathrm{tr}(X)$ is the usual trace of a matrix X.

(c) Deduce that if $G = D_{10}$, then G has at least two non-equivalent irreducible representations of degree two (equivalently two non-isomorphic two-dimensional simple $\mathbb{C}G$-modules).

6. Let A be a finite-dimensional algebra. A left ideal I of A is called *nilpotent* if there is some natural number $n \geq 1$ with $I^n = 0$, that is, such that $x_1 \cdots x_n = 0$ for all $x_i \in I$. Define the (Jacobson) *radical* of the algebra A as

$$\mathrm{rad}(A) = \{a \in A \mid a \cdot S = 0 \text{ for any simple } A\text{-module } S\}.$$

(a) Show that the sum of two nilpotent left ideals is nilpotent.

(b) Show that $\mathrm{rad}(A)$ is a two-sided ideal in A.

(c) By considering a composition series of A, or otherwise, show that $\mathrm{rad}(A)$ is nilpotent.

Conclude that the radical of an algebra A coincides with the largest nilpotent left ideal of A.

7. (a) Show that the only one-dimensional $\mathbb{C}S_n$-modules are the trivial module and the sign module. (The latter is the module in which each permutation acts by its signature.)

(b) Determine all the simple $\mathbb{C}S_3$-modules, up to isomorphism.

(c) A group representation $\rho : G \to GL(V)$ is called *faithful* if

$$\ker \rho = \{1\}.$$

Determine all the irreducible *non-faithful* representations of S_n (up to equivalence).