B2.1 Introduction to Representation Theory
Problem Sheet 3, MT 2019

The groups below are assumed to be finite and the representations finite-dimensional, unless stated otherwise.

1. Let V, W be two G-representations over $\mathbb{C}$. Prove that:
   (a) $\chi_{V\otimes W}(g) = \chi_V(g)\chi_W(g)$ for all $g \in G$;
   (b) $\chi_{V^*}(g) = \chi_V(g^{-1}) = \overline{\chi_V(g)}$ for all $g \in G$, where V^* denotes the representation contragredient to V.
   (c) Suppose W is a one-dimensional representation. Prove that $V \otimes W$ is irreducible if and only if V is irreducible.
   (d) Prove that V is irreducible if and only if V^* is irreducible.
   (e) Let St be the standard 3-dimensional representation of S_4. Decompose $\mathsf{St} \otimes \mathsf{St}$ into a direct sum of irreducible representations.

2. Let χ be the character of a $\mathbb{C}G$-module M. Show that $N = \{g \in G \mid \chi(g) = \chi(1)\}$ is a normal subgroup of G

   Deduce that a finite group G is simple if and only if $\chi(g) \neq \chi(1)$ for every $g \in G\backslash\{1\}$ and every nontrivial character of G.

3. Show that if two $\mathbb{C}G$-modules M_1 and M_2 have the same characters then they are isomorphic.

4. Let G act on a finite set Ω and let M be the permutation module with basis $\{e_w \mid w \in \Omega\}$ defined in lectures. Let $\chi = \chi_M$ be the character of M.

   Show that $\sum_{g\in G} \chi(g) = r|G|$ where r is the number of orbits of G on Ω.

   Suppose now that G is 2-transitive, that is G has two orbits acting on $\Omega \times \Omega$ in the action defined by $g \cdot (w_1, w_2) := (g \cdot w_1, g \cdot w_2)$

   Show that $\sum_{g\in G} \chi(g)^2 = 2|G|$ and deduce that M is a sum of two irreducible submodules $V_1 \oplus V_2$ where V_1 is the trivial module.

5. Find the character tables of Q_8 and D_8. Does the character table determine the group?

6. For a group G we denote by $[G, G]$ the subgroup generated by all elements $x^{-1}y^{-1}xy$ for all $x, y \in G$. The subgroup $[G, G]$ is the smallest normal subgroup N of G such that G/N is abelian.

   Let now G_1 and G_2 be two groups with the same character table. Show that $|G_1 : [G_1, G_1]| = |G_2 : [G_2, G_2]|$. Show further that the centre of G_1 has the same size as the centre of G_2.

7. Show that an element g of a finite group is conjugate to its inverse if and only if $\chi(g) \in \mathbb{R}$ for all characters of G.

[Optional for those who are taking Galois theory]: More generally show that $\chi(g) \in \mathbb{Z}$ if an only if g is conjugate to g^n for each integer n coprime to the order of g in G. What does this tell us about the character table of S_n?