B2.1 Introduction to Representation Theory
Problem Sheet 4, MT 2019

1. Find the character table of the alternating group A_5. (It may be helpful to remember that A_5 acts as the group rotations of the regular icosahedron. You may also think about restriction/induction between A_5 and S_5 if it helps.)

2. A conjugacy class g^G of a group G is called real if $g^G = (g^{-1})^G$ i.e., if g is conjugate to g^{-1}. A character χ of G is called real if $\chi(x) \in \mathbb{R}$ for all $x \in G$. Prove that the number of real conjugacy classes of a finite group is equal to the number of irreducible real characters. [Hint: Compute the dimension of the complex vector space

$$V := \{f : G \to \mathbb{C} \mid f(g) = f(h^{-1}gh) = f(g^{-1}) \quad \forall g, h \in G\}$$

in two different ways.

3. Let G be a finite group with an irreducible representation $\rho : G \to GL(2, C)$.

   (a) Prove that G has an element a of order 2.

   (b) For a as above show that either $\det \rho(a) \neq 1$ or else $\rho(a)$ is central in $GL(2, C)$.

   (c) Deduce that a finite simple group cannot have an irreducible representation of degree 2.

4. Prove that every finite group G has a faithful representation. Which finite abelian groups have a faithful *irreducible* representation?

5. Recall that an element e of an algebra A is called an idempotent if $e^2 = e$. Let G be a finite group and suppose V is a simple $\mathbb{C}G$-module. Define

$$e_V = \frac{\dim V}{|G|} \sum_{g \in G} \overline{\chi_V(g)} g \in \mathbb{C}G.$$

   (a) Prove that e_V is an element of the centre of $\mathbb{C}G$.

   (b) Let V' be a simple $\mathbb{C}G$-module. Prove that e_V acts on V' by 0 if $V' \ncong V$ and it acts by the identity on V.

   (c) Prove that if $\{V_i : 1 \leq i \leq n\}$ is the set of irreducible G-representations (up to isomorphism) and $e_i = e_{V_i}$, then $e_i^2 = e_i$ and $e_i \cdot e_j = 0$ in $\mathbb{C}G$. How does this relate to the Artin-Wedderburn Theorem?

6. Determine the restriction of the standard representation of S_4 to S_3. Compute the induced of the trivial representation of S_3 to S_4. Use this to illustrate Frobenius reciprocity.

7. Decompose into irreducible G-representations the induced representation $\mathrm{Ind}_H^G W$ where $G = S_4$ and

(a) $H = \langle(1234)\rangle$ and $W = \mathbb{C}v$ is the one-dimensional representation defined by $(1234) \cdot v = iv$, where $i = \sqrt{-1}$.

(b) $H = \langle(123)\rangle$ and $W = \mathbb{C}v$ is the one-dimensional representation defined by $(123) \cdot v = e^{2\pi i/3}v$.

8. (*optional*) Here is another result of Burnside: Let V be an irreducible representation of a finite group G and assume that $\dim V > 1$. Prove that χ_V takes the value 0 on some conjugacy class of G. (Hint: assume first that χ_V takes integer values.)

9. (*optional*) Suppose that V is a faithful representation of G. Show that every irreducible representation of G appears in some tensor power $V^{\otimes n} = V \otimes V \otimes \cdots \otimes V$ of V. (Hint: for an arbitrary irreducible character χ, consider the infinite series $\sum_{n\geq 0}\langle \chi, \chi_{V^{\otimes n}}\rangle_G \, t^n$, where t is an indeterminate.)

10. (*optional*) Which irreducible representations of S_n remain irreducible when restricted to A_n? Which irreducible representations of S_n are induced from A_n?