

B1.1 Logic

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Introduction

1. What is mathematical logic about?

- provide a uniform, unambiguous **language** for mathematics
- make precise what a **proof** is
- explain and guarantee **exactness, rigor and certainty** in mathematics
- establish the **foundations** of mathematics

$$\begin{aligned} & \text{B1 (Foundations)} \\ & = \text{B1.1 (Logic)} + \text{B1.2 (Set theory)} \end{aligned}$$

N.B.: Course does not teach you to think logically, but it explores what it *means* to think logically

2. Historical motivation

- *19th cent.:*
need for conceptual foundation in analysis:
what is the correct notion of
infinity, infinitesimal, limit, ...
- attempts to formalize mathematics:
 - *Frege's Begriffsschrift*
 - *Cantor's **naive** set theory:*
a set is any collection of objects
- led to **Russell's paradox:**
consider the set $R := \{S \text{ set} \mid S \notin S\}$
$$R \in R \Rightarrow R \notin R \text{ contradiction}$$
$$R \notin R \Rightarrow R \in R \text{ contradiction}$$

 \leadsto *fundamental crisis in the foundations of mathematics*

3. Hilbert's Program

1. find a uniform (formal) **language** for all mathematics
 2. find a complete system of **inference rules/ deduction rules**
 3. find a complete system of mathematical **axioms**
 4. prove that the system 1.+2.+3. is **consistent**, i.e. does not lead to contradictions
- ★ **complete:** every mathematical sentence can be proved or disproved using 2. and 3.
 - ★ 1., 2. and 3. should be **finitary/effective/computable/algorithmic** so, e.g., in 3. you can't take as axioms *the system of all true sentences in mathematics*
 - ★ **idea:** any piece of information is of finite length

4. Solutions to Hilbert's program

Step 1. is possible in the framework of
ZF = *Zermelo-Fraenkel set theory* or
ZFC = **ZF** + *Axiom of Choice*
(this is an empirical fact)
↷ B1.2 Set Theory HT 2020

Step 2. is possible in the framework of
1st-order logic:
Gödel's Completeness Theorem
↷ B1.1 Logic - this course

Step 3. is not possible (↷ C1.2):
Gödel's 1st Incompleteness Theorem:
there is no effective axiomatization
of arithmetic

Step 4. is not possible (↷ C1.2):
Gödel's 2nd Incompleteness Theorem

5. Decidability

Step 3. of Hilbert's program fails:

there is no effective axiomatization
for the entire body of mathematics

But: many important parts of mathematics
are completely and effectively axiomatizable,
they are **decidable**, i.e. there is an
algorithm = program = effective procedure
deciding whether a sentence is true or false
↪ allows proofs by computer

Example: $Th(\mathbb{C})$ = the **1st-order theory** of \mathbb{C}
= all *algebraic* properties of \mathbb{C} :

Axioms = *field axioms*

- + *all non-constant polynomials have a zero*
- + *the characteristic is 0*

Every algebraic property of \mathbb{C} follows from these
axioms.

Similarly for $Th(\mathbb{R})$.

↪ C1.1 Model Theory

6. Why *mathematical* logic?

1. Language and deduction rules are tailored for *mathematical objects* and mathematical ways of reasoning
N.B.: Logic tells you what a proof *is*, not how to *find* one
2. The *method* is mathematical:
we will develop logic as a *calculus* with sentences and formulas
⇒ Logic is itself a mathematical discipline, not meta-mathematics or philosophy, no ontological questions like *what is a number?*
3. Logic has *applications* towards other areas of mathematics, e.g. Algebra, Topology, but also towards theoretical computer science