10. Free and bound variables

Recall Example 9.5: The formula

$$\phi = \forall x_0 \forall x_1 (f(x_0, x_2) \doteq f(x_1, x_2) \to x_0 \doteq x_1)$$

- is true in $< \mathbf{Z}; \cdot >$ under any assignment v with $v(x_2) = 2$

- but false when $v(x_2) = 0$.

Whether or not $\mathcal{A} \models \phi[v]$ only depends on $v(x_2)$, not on $v(x_0)$ or $v(x_1)$.

say: the occurrence of x_2 in ϕ is **free**.

10.1 Definition

Let $\mathcal{L}$ be a first-order language, ϕ an $\mathcal{L}$-formula, and $x \in \{x_0, x_1, \ldots\}$ a variable occurring in ϕ.

The occurrence of x in ϕ is **free**, if
(i) ϕ is atomic, or
(ii) $\phi = \neg\psi$ resp. $\phi = (\chi \to \rho)$ and x occurs free in ψ resp. in χ or ρ, or
(iii) $\phi = \forall x_i \psi$, x occurs free in ψ, and $x \neq x_i$.

Every other occurrence of x in ϕ is called **bound**.

In particular, if $x = x_i$ and $\phi = \forall x_i \psi$, then x is bound in ϕ.

10.2 Example

$$(\exists x_0 P(\underbrace{x_0}_{b}, \underbrace{x_1}_{f}) \vee \forall x_1 (P(\underbrace{x_0}_{f}, \underbrace{x_1}_{b}) \to \exists x_0 P(\underbrace{x_0}_{b}, \underbrace{x_1}_{b})))$$

10.3 Lemma

Let $\mathcal{L}$ be a language, let $\mathcal{A}$ be an $\mathcal{L}$-structure, let v, v' be assignments in $\mathcal{A}$ and let ϕ be an $\mathcal{L}$-formula.

Suppose $v(x_i) = v'(x_i)$ for every variable x_i with a free *occurrence in ϕ.*

Then

$$\mathcal{A} \models \phi[v] \text{ iff } \mathcal{A} \models \phi[v'].$$

Proof:
For ϕ atomic: exercise

Now use induction on the length of ϕ:
- $\phi = \neg\psi$ and $\phi = (\chi \to \rho)$: easy
- $\phi = \forall x_i \psi$:

IH: Assume the Lemma holds for ψ.

Let
Free (ϕ):=$\{x_j \mid x_j$ occurs free in $\phi\}$
Free (ψ):=$\{x_j \mid x_j$ occurs free in $\psi\}$

$\Rightarrow x_i \notin$ Free(ϕ) and

$$\text{Free}(\phi) = \text{Free}(\psi) \setminus \{x_i\}$$

Assume $\mathcal{A} \models \forall x_i \psi[v]$ $(\star)$

to show: for any $v^\star$ agreeing with v' except possibly at x_i: $\mathcal{A} \models \psi[v^\star]$.

for all $x_j \in$ Free(ϕ):

$$v^\star(x_j) = v(x_j) = v'(x_j).$$

Let $v^+(x_j) := \begin{cases} v(x_j) & \text{if } j \neq i \\ v^\star(x_j) & \text{if } j = i \end{cases}$

Then v^+ agrees with v except possibly at x_i.

Hence, by $(\star)$, $\mathcal{A} \models \psi[v^+]$.

But $v^\star(x_j) = v^+(x_j)$ for all $x_j \in$ Free(ψ).

$\Rightarrow$ by IH, $\mathcal{A} \models \psi[v^\star]$ $\square$

10.4 Corollary
Let $\mathcal{L}$ be a language, $\alpha, \beta \in$ Form($\mathcal{L}$). Assume the variable x_i has no free occurrence in α. Then

$$\models (\forall x_i(\alpha \to \beta) \to (\alpha \to \forall x_i \beta)).$$

Proof:
Let $\mathcal{A}$ be an $\mathcal{L}$-structure and let v be an assignment in $\mathcal{A}$ such that
$\mathcal{A} \models \forall x_i(\alpha \to \beta)[v]$ $(\star)$

to show: $\mathcal{A} \models (\alpha \to \forall x_i \beta)[v]$.

So suppose $\mathcal{A} \models \alpha[v]$
to show: $\mathcal{A} \models \forall x_i \beta[v]$.

So let $v^\star$ be an assignment agreeing with v except possibly at x_i.
We want: $\mathcal{A} \models \beta[v^\star]$

x_i is *not* free in $\alpha \Rightarrow_{10.3} \mathcal{A} \models \alpha[v^\star]$
$(\star) \Rightarrow \mathcal{A} \models (\alpha \to \beta)[v^\star]$
$\Rightarrow \mathcal{A} \models \beta[v^\star]$ $\Box$

10.5 Definition
A formula ϕ without free (occurrence of) variables is called a **statement** or a **sentence**.

So for any $\mathcal{L}$-structure $\mathcal{A}$ and any assignment v in $\mathcal{A}$, whether or not $\mathcal{A} \models \phi[v]$ does not depend on v.

So we write $\mathcal{A} \models \phi$ if $\mathcal{A} \models \phi[v]$ for some/all v.

Say: ϕ is **true** in $\mathcal{A}$, or $\mathcal{A}$ is a **model** of ϕ.

($\rightsquigarrow$ 'Model Theory')

10.6 Example

Let $\mathcal{L} = \{f, c\}$ be a language, where f is a binary function symbol, and c is a constant symbol.

Consider the sentences (we write x, y, z instead of x_0, x_1, x_2)

$$\begin{aligned}
\phi_1 &: \ \forall x \forall y \forall z f(x, f(y, z)) \doteq f(f(x, y), z) \\
\phi_2 &: \ \forall x \exists y (f(x, y) \doteq c \wedge f(y, x) \doteq c) \\
\phi_3 &: \ \forall x (f(x, c) \doteq x \wedge f(c, x) \doteq x)
\end{aligned}$$

and let $\phi = \phi_1 \wedge \phi_2 \wedge \phi_3$.

Let $\mathcal{A} =< A; \circ; e >$ be an $\mathcal{L}$-structure (i.e. $\circ$ is an interpretation of f, and e is an interpretation of c.)

Then $\mathcal{A} \models \phi$ iff $\mathcal{A}$ is a group.

10.7 Example

Let $\mathcal{L} = \{E\}$ be a language with $E = P_i^{(2)}$ a binary relation symbol. Consider

$$\begin{aligned}
\chi_1 &: \ \forall x E(x,x) \\
\chi_2 &: \ \forall x \forall y (E(x,y) \leftrightarrow E(y,x)) \\
\chi_3 &: \ \forall x \forall y \forall z (E(x,y) \rightarrow (E(y,z) \rightarrow E(x,z)))
\end{aligned}$$

Then for any $\mathcal{L}$-structure $< A; R >$:

$< A; R > \models (\chi_1 \wedge \chi_2 \wedge \chi_3)$ iff

R is an equivalence relation on A.

Note: Most mathematical concepts can be captured by first-order formulas.

10.8 Example

Let P be a 2-place (i.e. binary) predicate symbol, $\mathcal{L} := \{P\}$. Consider the statements

ψ_1 : $\forall x \forall y (P(x,y) \dot\vee\ x \doteq y \dot\vee\ P(y,x))$
($\dot\vee$ means *either - or exclusively:*
$(\alpha \dot\vee\ \beta) :\Leftrightarrow ((\alpha \vee \beta) \wedge \neg(\alpha \wedge \beta)))$
ψ_2 : $\forall x \forall y \forall z ((P(x,y) \wedge P(y,z)) \rightarrow P(x,z))$
ψ_3 : $\forall x \forall z (P(x,z) \rightarrow \exists y (P(x,y) \wedge P(y,z)))$
ψ_4 : $\forall y \exists x \exists z (P(x,y) \wedge P(y,z))$

These are the axioms for a **dense linear order without endpoints**. Let $\psi = (\psi_1 \wedge \ldots \wedge \psi_4)$. Then $< \mathbf{Q}; < > \models \psi$ and $< \mathbf{R}; < > \models \psi$.

But: *'Completeness'* of $< \mathbf{R}; < >$ is not captured in 1st-order terms using the langauge $\mathcal{L}$, but rather in 2nd-order terms, where also quantification over *subsets*, rather than only over *elements* of $\mathbf{R}$ is used:

$$\forall A, B \subseteq \mathbf{R}((A \ll B) \rightarrow \exists c \in \mathbf{R}(A \lll \{c\} \lll B),$$

where $A \ll B$ means that $a < b$ for every $a \in A$ and every $b \in B$ etc.