11. Substitution

Goal: Given $\phi \in \text{Form}(\mathcal{L})$ and $x_i \in \text{Free}(\phi)$
- want to replace x_i by a term t to obtain a new formula $\phi[t/x_i]$
(read: *'ϕ with x_i replaced by t'*)
- should have $\{\forall x_i \phi\} \models \phi[t/x_i]$

11.1 Example

Let $\mathcal{L} = \{f; c\}$ and let ϕ be $\exists x_1 f(x_1) \doteq x_0$.

$\Rightarrow$ $\text{Free}(\phi) = \{x_0\}$ and
'$\forall x_0 \phi$', i.e. '$\forall x_0 \exists x_1 f(x_1) \doteq x_0$'
says that f is onto.

- if $t = c$ then $\phi[t/x_0]$ is $\exists x_1 f(x_1) \doteq c$
- but if $t = x_1$ then $\phi[t/x_0]$ is $\exists x_1 f(x_1) \doteq x_1$, stating the existence of a fixed point of f — *no good:* there are fixed point free onto functions, e.g. '$+1$' on $\mathbf{Z}$.

Problem: the variable x_1 in t has become unintentionally bound in the substitution.
To avoid this we define:

11.2 Definition
For $\phi \in$ Form($\mathcal{L}$), for any variable x_i (not necessarily in Free(ϕ)) and for any term $t \in$ Term($\mathcal{L}$), define the phrase

‘t **is free for** x_i **in** ϕ’

and the substitution

$\phi[t/x_i]$ (‘ϕ **with** x_i **replaced by** t’)

recursively as follows:

(i) if ϕ is atomic, then t is free for x_i in ϕ and $\phi[t/x_i]$ is the result of replacing *every* occurrence of x_i in ϕ by t.

(ii) if $\phi = \neg\psi$ then
t is free for x_i in ϕ iff t is free for x_i in ψ.
In this case, $\phi[t/x_i] = \neg\alpha$, where $\alpha = \psi[t/x_i]$.

(iii) if $\phi = (\psi \rightarrow \chi)$ then
t is free for x_i in ϕ iff
t is free for x_i in both ψ and χ.
In this case, $\phi[t/x_i] = (\alpha \rightarrow \beta)$,
where $\alpha = \psi[t/x_i]$ and $\beta = \chi[t/x_i]$.

(iv) if $\phi = \forall x_j \psi$ then
t is free for x_i in ϕ

if $i = j$ or

if $i \neq j$, and x_j does not occur in t,
and t is free for x_i in ψ.

In this case $\phi[t/x_i] = \begin{cases} \phi & \text{if } i = j \\ \forall x_j \alpha & \text{if } i \neq j, \end{cases}$
where $\alpha = \psi[t/x_i]$.

11.3 Example
Let $\mathcal{L} = \{f, g\}$ and let ϕ be $\exists x_1 f(x_1) \doteq x_0$.
$\Rightarrow g(x_0, x_2)$ is free for x_0 in ϕ
and $\phi[g(x_0, x_2)/x_0]$ is $\exists x_1 f(x_1) \doteq g(x_0, x_2)$,
but $g(x_0, x_1)$ is *not* free for x_0 in ϕ.

11.4 Lemma

Let $\mathcal{L}$ be a first-order language, $\mathcal{A}$ an $\mathcal{L}$-structure, $\phi \in$ Form($\mathcal{L}$) and t a term free for the variable x_i in ϕ. Let v be an assignment in $\mathcal{A}$ and define

$$v'(x_j) := \begin{cases} v(x_j) & \text{if } j \neq i \\ \widetilde{v}(t) & \text{if } j = i \end{cases}$$

Then $\mathcal{A} \models \phi[v']$ iff $\mathcal{A} \models \phi[t/x_i][v]$.

Proof: **1.** For $u \in$ Term($\mathcal{L}$) let

$u[t/x_i] :=$ the term obtained by replacing each occurrence of x_i in u by t

$\Rightarrow \widetilde{v'}(u) = \widetilde{v}(u[t/x_i])$
(Exercise)

2. If ϕ is **atomic**, say

$\phi = P(t_1, \ldots, t_k)$ for some $P = P_i^{(k)} \in \text{Pred}(\mathcal{L})$

then

$$\mathcal{A} \models \phi[v']$$

iff	$P_{\mathcal{A}}(\widetilde{v'}(t_1), \ldots, \widetilde{v'}(t_k))$	by def. '$\models$'
iff	$P_{\mathcal{A}}(\widetilde{v}(t_1[t/x_i]), \ldots, \widetilde{v}(t_k[t/x_i]))$	by 1.
iff	$\mathcal{A} \models P(t_1[t/x_i], \ldots, t_k[t/x_i])[v]$	by def. '$\models$'
iff	$\mathcal{A} \models \phi[t/x_i][v]$	

Similarly, if ϕ is $t_1 \doteq t_2$.

3. Induction step

The cases $\neg$ and $\rightarrow$ are routine.

$\leadsto$ the only interesting case is $\phi = \forall x_j \psi$.

IH: Lemma holds for ψ.

Case 1: $j = i$
$\Rightarrow \phi[t/x_i] = \phi$ by Definition 11.2.(iv)

$x_i = x_j \notin$ Free(ϕ)
$\Rightarrow v$ and v' agree on all $x \in$ Free(ϕ)
$\Rightarrow$ by Lemma 10.3,

$$\mathcal{A} \models \phi[v'] \text{ iff } \mathcal{A} \models \phi[v] \text{ iff } \mathcal{A} \models \phi[t/x_i][v]$$

Case 2: $j \neq i$
'$\Rightarrow$': Suppose $\mathcal{A} \models \forall x_j \psi[v']$ $\quad (\star)$

to show: $\mathcal{A} \models \forall x_j \psi[t/x_i][v]$

So let $v^\star$ agree with v except possibly at x_j.
to show: $\mathcal{A} \models \psi[t/x_i][v^\star]$

Define $v^{\star\prime}(x_k) := \begin{cases} v^\star(x_k) & \text{if } k \neq i \\ \widetilde{v^\star}(t) & \text{if } k = i \end{cases}$
t is free for x_i in $\phi \Rightarrow$
t is free for x_i in ψ and t does not contain x_j.

IH $\Rightarrow$ enough to show: $\mathcal{A} \models \psi[v^{\star\prime}]$

$v^{\star\prime}$ and v' agree except possibly at x_i and x_j.
But, in fact, they *do* agree at x_i:

$$v'(x_i) = \tilde{v}(t) = \widetilde{v^\star}(t) = v^{\star\prime}(x_i),$$

where the 2nd equality holds, because v and $v^\star$ agree except possibly at x_i, which does not occur in t.
So $v^{\star\prime}$ and v' agree except possibly at x_j
$\Rightarrow$ by $(\star)$, $\mathcal{A} \models \psi[v^{\star\prime}]$ as required.

'$\Leftarrow$': similar. $\square$

11.5 Corollary
For any $\phi \in$ Form($\mathcal{L}$), $t \in$ Term($\mathcal{L}$),

$$\models (\forall x_i \phi \rightarrow \phi[t/x_i]),$$

provided that the term t is free for x_i in ϕ.

Proof: Let $\mathcal{A}$ be an $\mathcal{L}$-structure and let v be an assignment in $\mathcal{A}$.

Assume $\mathcal{A} \models \forall x_i \phi[v]$ $\quad (\star)$
to show: $\mathcal{A} \models \phi[t/x_i][v]$

By Lemma 11.4, it suffices to show $\mathcal{A} \models \phi[v']$, where

$$v'(x_j) := \begin{cases} v(x_j) & \text{for } j \neq i \\ \widetilde{v}(t) & \text{for } j = i. \end{cases}$$

Since v and v' agree except possibly at x_i, this follows from $(\star)$.

□