12. A formal system for Predicate Calculus

12.1 Definition
Associate to each first-order language $\mathcal{L}$ the formal system $K(\mathcal{L})$ with the following axioms and rules (for any $\alpha, \beta, \gamma \in$ Form($\mathcal{L}$), $t \in$ Term($\mathcal{L}$)):

Axioms
A1 $(\alpha \to (\beta \to \alpha))$
A2 $((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma)))$
A3 $((\neg\beta \to \neg\alpha) \to (\alpha \to \beta))$
A4 $(\forall x_i \alpha \to \alpha[t/x_i])$, where t is free for x_i in α
A5 $(\forall x_i(\alpha \to \beta) \to (\alpha \to \forall x_i \beta))$, provided that $x_i \notin$ Free(α)
A6 $\forall x_i\, x_i \doteq x_i$
A7 $(x_i \doteq x_j \to (\phi \to \phi'))$, where ϕ is *atomic* and ϕ' is obtained from ϕ by replacing some (not necessarily all) occurrences of x_i in ϕ by x_j

Rules

MP (Modus Ponens) From α and $(\alpha \to \beta)$ infer β

$\forall$ **(Generalisation)** From α infer $\forall x_i \alpha$

Thinning Rule see 12.6

ϕ is a **theorem of** $K(\mathcal{L})$ (write '$\vdash \phi$') if there is a sequence (a **derivation**, or a **proof**) $\phi_1, \ldots, \phi_n$ of $\mathcal{L}$-formulas with $\phi_n = \phi$ such that each ϕ_i either is an axiom or is obtained from earlier ϕ_j's by MP or $\forall$.

For $\Gamma \subseteq \mathrm{Form}(\mathcal{L})$, $\phi \in \mathrm{Form}(\mathcal{L})$ define similarly that ϕ is **derivable in** $K(\mathcal{L})$ **from the hypotheses** Γ (write '$\Gamma \vdash \phi$'), except that the ϕ_i's may now also be formulas from Γ,
but we make the restriction that $\forall$ may only be used for variables x_i *not occurring free in any formula in* Γ.

12.2 Soundness Theorem for Pred. Calc.

If $\Gamma \vdash \phi$ then $\Gamma \models \phi$.

Proof: Induction on length of derivation

Clear that **A1**, **A2**, and **A3** are logically valid. So are **A4** and **A5** by Cor. 11.5 resp. Cor. 10.4.
Also **A6** is logically valid: easy exercise.

A7: Let $\mathcal{A}$ be an $\mathcal{L}$-structure and let v be any assignment in $\mathcal{A}$. Suppose that

$$\mathcal{A} \models x_i \doteq x_j[v] \text{ and } \mathcal{A} \models \phi[v].$$

We want to show that $\mathcal{A} \models \phi'[v]$ (with ϕ atomic).

Now $v(x_i) = v(x_j)$
$\Rightarrow \widetilde{v}(t') = \widetilde{v}(t)$ for any term t' obtained from t by replacing some of the x_i by x_j
(easy induction on terms)

If ϕ is $P(t_1, \ldots, t_k)$ then ϕ' is $P(t'_1, \ldots, t'_k)$.

$$\begin{array}{lll} \mathcal{A} \models \phi[v] & \text{iff} & P_{\mathcal{A}}(\tilde{v}(t_1), \ldots, \tilde{v}(t_k)) \\ & \text{iff} & P_{\mathcal{A}}(\tilde{v}(t'_1), \ldots, \tilde{v}(t'_k)) \\ & \text{iff} & \mathcal{A} \models P(t'_1, \ldots, t'_k)[v] \\ & \text{iff} & \mathcal{A} \models \phi'[v] \text{ as required} \end{array}$$

Similarly, if ϕ is $t_1 \doteq t_2$.
So now all axioms are logically valid.

MP is sound: for any $\mathcal{A}$, v

$\mathcal{A} \models \alpha\ [v]$ and $\mathcal{A} \models (\alpha \to \beta)[v]$ imply $\mathcal{A} \models \beta[v]$

Generalisation: IH for any $\mathcal{A}$, v
if $\mathcal{A} \models \psi[v]$ for all $\psi \in \Gamma$ then $\mathcal{A} \models \alpha[v]$ $\qquad (\star)$

to show: $\mathcal{A} \models \forall x_i \alpha[v]$ for such $\mathcal{A}$, v.

So let $v^\star$ agree with v except possibly at x_i.
$x_i \notin \text{Free}(\psi)$ for any $\psi \in \Gamma$
$\Rightarrow \mathcal{A} \models \psi[v^\star]$ for all $\psi \in \Gamma$ (by Lemma 10.3)
$\Rightarrow \mathcal{A} \models \alpha[v^\star]$ (by $(\star)$)
$\Rightarrow \mathcal{A} \models \forall x_i \alpha[v]$ as required. $\square$

12.3 Deduction Theorem for Pred. Calc.

If $\Gamma \cup \{\psi\} \vdash \phi$ *then* $\Gamma \vdash (\psi \to \phi)$.

Proof: same as for prop. calc. (Theorem 6.6) with one more step in the induction (on the length of the derivation).

IH: $\Gamma \vdash (\psi \to \phi_j)$
to show: $\Gamma \vdash (\psi \to \forall x_i \phi_j)$,
where generalisation ($\forall$) has been used to infer $\forall x_i \phi_j$ under the hypotheses $\Gamma \cup \{\psi\}$

$\Rightarrow$ $x_i \notin$ Free(γ) for any $\gamma \in \Gamma$ and $x_i \notin$ Free(ψ)
$\Rightarrow$ by IH and $\forall$: $\Gamma \vdash \forall x_i(\psi \to \phi_j)$
A5 $\vdash (\forall x_i(\psi \to \phi_j) \to (\psi \to \forall x_i \phi_j))$, since $x_i \notin$ Free(ψ)
$\Rightarrow$ by **MP**, $\Gamma \vdash (\psi \to \forall x_i \phi_j)$ as required.

$\square$

12.4 Tautologies

If A is a tautology of the *Propositional Calculus* with propositional variables among $p_0, \ldots, p_n$, and if $\psi_0, \ldots, \psi_n \in \mathrm{Form}(\mathcal{L})$ are formulas of *Predicate Calculus*, then the formula A' obtained from A by replacing each p_i by ψ_i is a **tautology of** $\mathcal{L}$:

Since **A1**, **A2**, **A3** and **MP** are in $K(\mathcal{L})$, one also has $\vdash A'$ in $K(\mathcal{L})$.

May use the tautologies in derivations in $K(\mathcal{L})$.

12.5 Example *Swapping variables*

Suppose x_j does not occur in ϕ.
Then $\{\forall x_i \phi\} \vdash \forall x_j \phi[x_j/x_i]$

1	$\forall x_i \phi$	[$\in \Gamma$]
2	$(\forall x_i \phi \to \phi[x_j/x_i])$	[A4]
3	$\phi[x_j/x_i]$	[MP 1,2]
4	$\forall x_j \phi[x_j/x_i]$	[$\forall$]

where $\forall$ may be applied in line 4, since x_j does not occur in ϕ.

This proof would not work if
$\Gamma = \{\forall x_i \phi, x_j \doteq x_j\}$ (say). Hence need (besides **MP** and ($\forall$))

12.6 Thinning Rule

If $\Gamma \vdash \phi$ and $\Gamma' \supseteq \Gamma$ then $\Gamma' \vdash \phi$.

12.7 Example

$$(\exists x_i \phi \to \psi) \vdash \forall x_i (\phi \to \psi),$$

where $x_i \notin$ Free(ψ).

Proof: Let $\Gamma = \{(\exists x_i \phi \to \psi), \neg\psi\}$

1	$(\neg\forall x_i \neg\phi \to \psi)$	[$\in \Gamma$]
2	$((\neg\forall x_i \neg\phi \to \psi) \to (\neg\psi \to \forall x_i \neg\phi))$	[taut.]
3	$(\neg\psi \to \forall x_i \neg\phi)$	[MP 1,2]
4	$\neg\psi$	[$\in \Gamma$]
5	$\forall x_i \neg\phi$	[MP 3,4]
6	$(\forall x_i \neg\phi \to \neg\phi)$	[A4]
7	$\neg\phi$	[MP 5,6]

Note that in line 6, x_i is free for x_i in ϕ.

Hence $\Gamma \vdash \neg\phi$. So

$$\begin{array}{ll} (\exists x_i \phi \to \psi) \vdash (\neg\psi \to \neg\phi) & \text{[DT]} \\ (\exists x_i \phi \to \psi) \vdash (\phi \to \psi) & \text{[A3, MP]} \\ (\exists x_i \phi \to \psi) \vdash \forall x_i (\phi \to \psi) & [\forall] \end{array}$$

$\square$