B1.1 Logic

MT 19

Problem Sheet $\sharp$ 4

(1)(i) Let $\mathcal{L}$ be a first-order language, let $\mathcal{A}$ be an $\mathcal{L}$-structure, let v be an assignment in $\mathcal{A}$ and let u and t be terms in $\mathcal{L}$. Define a new assignment v' by

$$v'(x_j) := \begin{cases} v(x_j) & \text{if } j \neq i \\ \widetilde{v}(t) & \text{if } j = i \end{cases}$$

Let $u[t/x_i]$ be the term obtained by replacing each occurrence of x_i in u by t. Show that then $\widetilde{v'}(u) = \widetilde{v}(u[t/x_i])$.

(ii) Prove that for any closed (i.e. variable free) terms t_1, t_2, t_3 one has $\vdash (t_1 \doteq t_2 \to t_2 \doteq t_1)$ and $\{t_1 \doteq t_2, t_2 \doteq t_3\} \vdash t_1 \doteq t_3$.

(2)(i) Prove that $\vdash (\forall x_i(A \to B) \to (\exists x_i A \to B))$ for any formulas A, B provided that the variable x_i does not occur free in B.

(ii) Let ϕ be a formula with just one variable, x_i say, occurring free and let Δ be a set of sentences. Assume that the constant symbol c_j does not occur in ϕ nor in any sentence in Δ, and that $\Delta \vdash \phi[c_j/x_i]$. Sketch a proof that $\Delta \vdash \phi$.

[Hint: First reduce to the case that Δ is finite and choose m so large that the variable x_m does not occur in Δ nor in any formula in a derivation of $\phi[c_j/x_i]$ from hyptheses Δ. Then change every occurrence of c_j to x_m.]

(3) Derive the following theorems:

(i) If x_i does not occur free in A then
(a) $\vdash (\exists x_i(A \to B) \to (A \to \exists x_i B))$,
(b) $\vdash ((A \to \exists x_i B) \to \exists x_i(A \to B))$.

(ii) If the only variables occurring free in ϕ are x_i and x_j (where $i \neq j$) then
(a) $\vdash (\forall x_i \neg\phi \to \neg\forall x_j \phi)$
(b) $\vdash (\exists x_i \forall x_j \phi \to \forall x_j \exists x_i \phi)$.

p.t.o.

(4) Let f, g be binary function symbols, P a binary predicate symbol, c, d constant symbols and let $\mathcal{L} := \{f, g; P; c, d\}$.

Consider $\mathcal{R} :=< \mathbb{R}; +, \cdot; <; 0, 1 >$ as $\mathcal{L}$-structure. Let h be a unary function symbol, let $\mathcal{L}' := \mathcal{L} \cup \{h\}$ and let $\mathcal{R}'$ be $\mathcal{R}$ together with an interpretation $h_{\mathcal{R}'}$ of h in $\mathcal{R}$.

Find $\mathcal{L}'$-formulas ϕ and ψ such that
(i) $\mathcal{R}' \models \phi$ iff $h_{\mathcal{R}'}$ is continuous.
(ii) $\mathcal{R}' \models \psi$ iff $h_{\mathcal{R}'}$ is differentiable.

(5)(i) Let $\mathcal{L} := \{+, \cdot; 0, 1\}$ be the language of rings (i.e. $+$ a binary function symbol etc.). Write down sets of formulas Φ_p (for p a prime or $p = 0$) whose models are exactly all fields of characteristic p.

(ii) State the Compactness Theorem for sets of sentences and show how it follows from the Soundness and Completeness Theorems.

(iii) Prove that Φ_0 in (i) cannot be chosen finite.

(6)(i) Axiomatize the first-order theory Σ of ordered fields in the language $\mathcal{L} := \{+, \cdot; <; 0, 1\}$.

(ii) Which of the following is a model of Σ:
(α) $\mathbb{Q}$ with the usual interpretations
(β) $\mathbb{R}$ with the usual interpretations
(γ) $\mathbb{C}$ with $a + bi < c + di$ iff $a^2 + b^2 < c^2 + d^2$
(δ) $\mathbb{F}_p$, the field with p (prime) elements with $0 < 1 < 2 < \cdots < p - 1$.

(iii) Is Σ consistent? Is it maximal consistent?

(iv) Recall that the ordering on $\mathbb{Q}$ resp. on $\mathbb{R}$ is *archimedean*, i.e. for every $x \in \mathbb{Q}$ resp. $x \in \mathbb{R}$ there is some $n \in \mathbb{N}$ with $-n < x < n$. Use the Compactness Theorem to prove that archimedeanity is not a first-order property. (Hint: introduce a new constant symbol c.)