

13.7 Theorem (CT after reduction 13.4)

Let Γ be a maximal consistent witnessing set of sentences not containing a \doteq -symbol.

Then Γ has a model.

Proof:

Let $A := \{t \in \text{Term}(\mathcal{L}) \mid t \text{ is closed}\}$

(recall: t **closed** means no variables in t).

A will be the domain of our model \mathcal{A} of Γ
(\mathcal{A} is called **term model**).

For $P = P_n^{(k)} \in \text{Pred}(\mathcal{L})$ resp. $f = f_n^{(k)} \in \text{Fct}(\mathcal{L})$ resp. $c = c_n \in \text{Const}(\mathcal{L})$ define the interpretations $P_{\mathcal{A}}$ resp. $f_{\mathcal{A}}$ resp. $c_{\mathcal{A}}$ by

$$\begin{aligned} P_{\mathcal{A}}(t_1, \dots, t_k) \text{ holds} & :\Leftrightarrow \Gamma \vdash P(t_1, \dots, t_k) \\ f_{\mathcal{A}}(t_1, \dots, t_k) & := f(t_1, \dots, t_k) \\ c_{\mathcal{A}} & := c \end{aligned}$$

to show: $\mathcal{A} \models \Gamma$

(i.e. $\mathcal{A} \models \Gamma[v]$ for some/all assignments v in \mathcal{A} : note that Γ contains only sentences).

Let v be an assignment in \mathcal{A} ,
say $v(x_i) =: s_i \in A$ for $i = 0, 1, 2, \dots$

Claim 1: For any $u \in \text{Term}(\mathcal{L})$: $\tilde{v}(u) = u[\vec{s}/\vec{x}]$
(:= the closed term obtained by replacing each x_i in u by s_i)

Proof: by induction on u

- $u = x_i \Rightarrow$

$$\tilde{v}(u) = v(x_i) = s_i = x_i[s_i/x_i] = u[\vec{s}/\vec{x}]$$

- $u = c \in \text{Const}(\mathcal{L}) \Rightarrow$

$$\tilde{v}(u[\vec{s}/\vec{x}]) = \tilde{v}(u) = \tilde{v}(c) = c_{\mathcal{A}}$$

- $u = f(t_1, \dots, t_k) \Rightarrow$

$$\begin{aligned} \tilde{v}(u) &:= f_{\mathcal{A}}(\tilde{v}(t_1), \dots, \tilde{v}(t_k)) \\ &= f_{\mathcal{A}}(t_1[\vec{s}/\vec{x}], \dots, t_k[\vec{s}/\vec{x}]) && \text{by IH} \\ &= f(t_1[\vec{s}/\vec{x}], \dots, t_k[\vec{s}/\vec{x}]) && \text{by def. of } f_{\mathcal{A}} \\ &= f(t_1, \dots, t_k)[\vec{s}/\vec{x}] && \text{by def. of subst.} \\ &= u[\vec{s}/\vec{x}] && \square \text{Claim 1} \end{aligned}$$

Claim 2: For any $\phi \in \text{Form}(\mathcal{L})$ without \doteq -symbol:

$$\mathcal{A} \models \phi[v] \text{ iff } \Gamma \vdash \phi[\vec{s}/\vec{x}],$$

where $\phi[\vec{s}/\vec{x}] :=$ the sentence obtained by replacing each *free* occurrence of x_i by s_i : note that s_i is free for x_i in ϕ because s_i is a *closed* term.

Proof: by induction on ϕ

ϕ **atomic**, i.e.

$\phi = P(t_1, \dots, t_k)$ for some $P = P_n^{(k)} \in \text{Pred}(\mathcal{L})$

Then

$$\begin{aligned} & \mathcal{A} \models \phi[v] \\ \text{iff } & P_{\mathcal{A}}(\tilde{v}(t_1), \dots, \tilde{v}(t_k)) && \text{[def. of '}\models\text{'}] \\ \text{iff } & P_{\mathcal{A}}(t_1[\vec{s}/\vec{x}], \dots, t_k[\vec{s}/\vec{x}]) && \text{[Claim 1]} \\ \text{iff } & \Gamma \vdash P(t_1[\vec{s}/\vec{x}], \dots, t_k[\vec{s}/\vec{x}]) && \text{[def. of } P_{\mathcal{A}}\text{]} \\ \text{iff } & \Gamma \vdash P(t_1, \dots, t_k)[\vec{s}/\vec{x}] && \text{[def. subst.]} \\ \text{iff } & \Gamma \vdash \phi[\vec{s}/\vec{x}] \end{aligned}$$

Note that Claim 2 might be false for formulas of the form $t_1 \doteq t_2$: might have $\Gamma \vdash c_0 \doteq c_1$, but c_0, c_1 are distinct elements in A .

Induction Step

$$\begin{aligned} & \mathcal{A} \models \neg\phi[v] \\ \text{iff } & \text{not } \mathcal{A} \models \phi[v] && [\text{def. of '}\models\text{'}] \\ \text{iff } & \text{not } \Gamma \vdash \phi[\vec{s}/\vec{x}] && [\text{IH}] \\ \text{iff } & \Gamma \vdash \neg\phi[\vec{s}/\vec{x}] && [\Gamma \text{ max. cons.}] \end{aligned}$$

$$\begin{aligned} & \mathcal{A} \models (\phi \rightarrow \psi)[v] \\ \text{iff } & \text{not } \mathcal{A} \models \phi[v] \text{ or } \mathcal{A} \models \psi[v] && [\text{def. '}\models\text{'}] \\ \text{iff } & \text{not } \Gamma \vdash \phi[\vec{s}/\vec{x}] \text{ or } \Gamma \vdash \psi[\vec{s}/\vec{x}] && [\text{IH}] \\ \text{iff } & \Gamma \vdash \neg\phi[\vec{s}/\vec{x}] \text{ or } \Gamma \vdash \psi[\vec{s}/\vec{x}] && [\Gamma \text{ max.}] \\ \text{iff } & \Gamma \vdash (\neg\phi[\vec{s}/\vec{x}] \vee \psi[\vec{s}/\vec{x}]) && [\text{def. '}\vdash\text{'}] \\ \text{iff } & \Gamma \vdash (\phi[\vec{s}/\vec{x}] \rightarrow \psi[\vec{s}/\vec{x}]) && [\text{taut.}] \\ \text{iff } & \Gamma \vdash (\phi \rightarrow \psi)[\vec{s}/\vec{x}] && [\text{def. subst.}] \end{aligned}$$

\forall -step ' \Rightarrow '

Suppose $\mathcal{A} \models \forall x_i \phi[v]$ (★)

but not $\Gamma \vdash (\forall x_i \phi)[\vec{s}/\vec{x}]$

$\Rightarrow \Gamma \vdash (\neg\forall x_i \phi)[\vec{s}/\vec{x}]$ (Γ max.)

$\Rightarrow \Gamma \vdash (\exists x_i \neg\phi)[\vec{s}/\vec{x}]$ (Exercise)

Now let ϕ' be the result of substituting each free occurrence of x_j in ϕ by s_j for all $j \neq i$.

$$\Rightarrow (\exists x_i \neg \phi)[\vec{s}/\vec{x}] = \exists x_i \neg \phi'$$

$$\Rightarrow \Gamma \vdash \exists x_i \neg \phi'$$

Γ witnessing \Rightarrow

$$\Gamma \vdash \neg \phi'[c/x_i] \text{ for some } c \in \text{Const}(\mathcal{L})$$

Define

$$v^*(x_j) := \begin{cases} v(x_j) & \text{if } j \neq i \\ c & \text{if } j = i \end{cases} \quad \text{and} \quad s_j^* := \begin{cases} s_j & \text{if } j \neq i \\ c & \text{if } j = i \end{cases}$$

$$\Rightarrow \neg \phi'[c/x_i] = \neg \phi[s^*/\vec{x}]$$

$$\Rightarrow \Gamma \vdash \neg \phi[s^*/\vec{x}]$$

$$\Rightarrow \Gamma \models \neg \phi[v^*] \quad \text{[IH]}$$

But, by (\star) , $\mathcal{A} \models \phi[v^*]$: contradiction.

\forall -step ‘ \Leftarrow ’:

Suppose $\mathcal{A} \not\models \forall x_i \phi[v]$

\Rightarrow for some v^* agreeing with v except possibly at x_i

$$\mathcal{A} \models \neg \phi[v^*]$$

$$\text{Let } s_j^* := \begin{cases} s_j & \text{for } j \neq i \\ v^*(x_j) & \text{for } j = i \end{cases}$$

$$\text{IH } \Rightarrow \Gamma \vdash \neg \phi[\vec{s}^*/\vec{x}],$$

$$\text{i.e. } \Gamma \vdash \neg \phi'[s_i^*/x_i],$$

where ϕ' is the result of substituting each free occurrence of x_j in ϕ by s_j for all $j \neq i$

$$\Rightarrow \Gamma \vdash \exists x_i \neg \phi'$$

(Exercise:

$\chi \in \text{Form}(\mathcal{L})$, $\text{Free}(\chi) \subseteq \{x_i\}$, s a closed term

$$\Rightarrow \vdash (\chi[s/x_i] \rightarrow \exists x_i \chi))$$

So

$$\begin{aligned} & \Gamma \vdash \neg \forall x_i \neg \neg \phi' \\ \Rightarrow & \Gamma \vdash \neg \forall x_i \phi' \\ \Rightarrow & \Gamma \vdash (\neg \forall x_i \phi)[\vec{s}/\vec{x}] \\ \Rightarrow & \Gamma \not\vdash (\forall x_i \phi)[\vec{s}/\vec{x}] \end{aligned}$$

□ Claim 2

Now choose any $\phi \in \Gamma \subseteq \text{Sent}(\mathcal{L})$

$$\Rightarrow \phi[\vec{s}/\vec{x}] = \phi$$

$$\Rightarrow \mathcal{A} \models \phi[v], \text{ i.e. } \mathcal{A} \models \phi \quad [\text{Claim 2}]$$

$$\Rightarrow \mathcal{A} \models \Gamma$$

□13.7

13.8 Modification required for \doteq -symbol

Define an equivalence relation E on A by

$$t_1 E t_2 \text{ iff } \Gamma \vdash t_1 \doteq t_2$$

(easy to check: this *is* an equivalence relation, e.g. transitivity = (1)(ii) of sheet $\#$ 4).

Let A/E be the set of equivalence classes t/E (with $t \in A$).

Define \mathcal{L} -structure \mathcal{A}/E with domain A/E by

$$\begin{aligned} P_{\mathcal{A}/E}(t_1/E, \dots, t_k/E) & :\Leftrightarrow \Gamma \vdash P(t_1, \dots, t_k) \\ f_{\mathcal{A}/E}(t_1/E, \dots, t_k/E) & := f_{\mathcal{A}}(t_1, \dots, t_k)/E \\ c_{\mathcal{A}/E} & := c_{\mathcal{A}}/E \end{aligned}$$

check: independence of representatives of t/E (this is the purpose of Axiom **A7**).

Rest of the proof is much the same as before.

□13.1