14. Applications of Gödel's Completeness Theorem

14.1 Compactness Theorem for Predicate Calculus

Let $\mathcal{L}$ be a first-order language and let $\Gamma \subseteq Sent(\mathcal{L})$.

Then Γ has a model iff every finite subset of Γ has a model.

Proof: as for Propositional Calculus – Exercise sheet $\sharp$ 4, (5)(ii).

14.2 Example

Let $\Gamma \subseteq Sent(\mathcal{L})$. Assume that for every $N \geq 1$, Γ has a model whose domain has at least N elements.

Then Γ has a model with an infinite domain.

Proof:

For each $n \geq 2$ let χ_n be the sentence

$$\exists x_1 \exists x_2 \cdots \exists x_n \bigwedge_{1 \leq i < j \leq n} \neg x_i \doteq x_j$$

$\Rightarrow$ for any $\mathcal{L}$-structure $\mathcal{A} = \langle A; \ldots \rangle$,

$$\mathcal{A} \models \chi_n \text{ iff } \sharp A \geq n$$

Let $\Gamma' := \Gamma \cup \{\chi_n \mid n \geq 1\}$.

If $\Gamma_0 \subseteq \Gamma'$ is finite,
let N be maximal with $\chi_N \in \Gamma_0$.
By hypothesis, $\Gamma \cup \{\chi_N\}$ has a model.
$\Rightarrow$ Γ_0 has a model
(note that $\vdash \chi_N \to \chi_{N-1} \to \chi_{N-2} \to \ldots$)

$\Rightarrow$ By the Compactness Theorem 14.1,
Γ' has a model, say $\mathcal{A} = \langle A; \ldots \rangle$

$\Rightarrow$ $\mathcal{A} \models \chi_n$ for all $n \Rightarrow$ $\sharp A = \infty$ $\square$

14.3 The Löwenheim-Skolem Theorem
Let $\Gamma \subseteq Sent(\mathcal{L})$ be consistent.
Then Γ has a model with a countable domain.

Proof:
This follows from the proof of the Completeness Theorem:
The **term model** constructed there was countable, because there are only countably many closed terms.

$\square$

14.4 Definition
(i) Let $\mathcal{A}$ be an $\mathcal{L}$-structure.
Then the **$\mathcal{L}$-theory of $\mathcal{A}$** is

$$\mathrm{Th}(\mathcal{A}) := \{\phi \in \mathrm{Sent}(\mathcal{L}) \mid \mathcal{A} \models \phi\},$$

the set of all $\mathcal{L}$-sentences true in $\mathcal{A}$.
Note: $\mathrm{Th}(\mathcal{A})$ is maximal consistent.
(ii) If $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{L}$-structures with $\mathrm{Th}(\mathcal{A}) = \mathrm{Th}(\mathcal{B})$ then $\mathcal{A}$ and $\mathcal{B}$ are **elementarily equivalent** (in symbols '$\mathcal{A} \equiv \mathcal{B}$').

14.5 Remark
Let $\Gamma \subseteq$*Sent*$(\mathcal{L})$ *be any set of* $\mathcal{L}$*-sentences.*

(a) *Then TFAE:*

(i) Γ *is strongly maximal consistent (i.e. for each* $\mathcal{L}$*-sentence* ϕ*,* $\phi \in \Gamma$ *of* $\neg\phi \in \Gamma$*)*

(ii) $\Gamma = \mathrm{Th}(\mathcal{A})$ *for some* $\mathcal{L}$*-structure* $\mathcal{A}$

Proof:
(i) $\Rightarrow$ **(ii):** Completeness Theorem
Rest: clear. $\square$

(b) Γ *is maximal consistent if and only if* Γ *has models, and, for any two models* $\mathcal{A}$ *and* $\mathcal{B}$*,* $\mathcal{A} \equiv \mathcal{B}$*.*

A worked example: Dense linear orderings without endpoints

Let $\mathcal{L} = \{<\}$ be the language with just one binary predicate symbol '$<$',
and let Γ be the $\mathcal{L}$-theory of dense linear orderings without endpoints (cf. Example 10.8) consisting of the axioms $\psi_1, \ldots, \psi_4$:

$$\begin{aligned}
\psi_1 :&\ \forall x \forall y ((x < y \vee x \doteq y \vee y < x) \\
&\ \wedge \neg((x < y \wedge x \doteq y) \vee (x < y \wedge y < x))) \\
\psi_2 :&\ \forall x \forall y \forall z (x < y \wedge y < z) \to x < z) \\
\psi_3 :&\ \forall x \forall z (x < z \to \exists y (x < y \wedge y < z)) \\
\psi_4 :&\ \forall y \exists x \exists z (x < y \wedge y < z)
\end{aligned}$$

14.6 (a) Examples
$\mathbb{Q}$, $\mathbb{R}$, $]0,1[$, $\mathbb{R} \setminus \{0\}$, $[\sqrt{2}, \pi] \cap \mathbb{Q}$, $]0,1[\cup]2,3[$, or $\mathbb{Z} \times \mathbb{R}$ with lexicographic ordering:
$(a,b) < (c,d) \Leftrightarrow a < c$ or $(a = c \;\&\; b < d)$

(b) Counterexamples $[0,1]$, $\mathbb{Z}$, $\{0\}$, $\mathbb{R} \setminus]0,1[$ or $\mathbb{R} \times \mathbb{Z}$ with lexicographic ordering

14.7 Theorem

Let Γ be the theory of dense linear orderings without endpoints, and let $\mathcal{A} = \langle A; <_{\mathcal{A}} \rangle$ and $\mathcal{B} = \langle B; <_{\mathcal{B}} \rangle$ be two countable models.
Then $\mathcal{A}$ and $\mathcal{B}$ are isomorphic, i.e. there is an order preserving bijection between A and B.

Proof: Note: A and B are infinite.
Choose an enumeration (no repeats)

$$\begin{aligned} A &= \{a_1, a_2, a_3, \ldots\} \\ B &= \{b_1, b_2, b_3, \ldots\} \end{aligned}$$

Define $\phi : A \to B$ recursively s.t. for all n:

($\star_n$) for all $i, j \leq n :\ \phi(a_i) <_{\mathcal{B}} \phi(a_j) \Leftrightarrow a_i <_{\mathcal{A}} a_j$

Suppose ϕ has been defined on $\{a_1, \ldots, a_n\}$ satisfying ($\star_n$).

Let $\phi(a_{n+1}) = b_m$,
where $m > 1$ is minimal s.t.

$$\text{for all } i \leq n: \; b_m <_{\mathcal{B}} \phi(a_i) \Leftrightarrow a_{n+1} <_{\mathcal{A}} a_i,$$

i.e. the position of $\phi(a_{n+1})$
relative to $\phi(a_1), \ldots, \phi(a_n)$

is the same as that of a_{n+1}
relative to $a_1, \ldots, a_n$

(possible as $\mathcal{A}, \mathcal{B} \models \Gamma$).

$\Rightarrow$ $(\star_{n+1})$ holds for $a_1, \ldots, a_{n+1}$

$\Rightarrow$ ϕ is injective

And ϕ is surjective, by minimality of m. $\Box$

14.8 Corollary
Γ is maximal consistent

Proof:
to show: Th($\mathcal{A}$) =Th($\mathcal{B}$) for any $\mathcal{A}, \mathcal{B} \models \Gamma$
(by Remark 14.5(b))

By the Theorem of Löwenheim-Skolem (14.3), Th($\mathcal{A}$) and Th($\mathcal{B}$) have countable models, say $\mathcal{A}_0$ and $\mathcal{B}_0$.

$\Rightarrow$ Th($\mathcal{A}_0$) =Th($\mathcal{A}$) and Th($\mathcal{B}_0$) =Th($\mathcal{B}$)

Theorem 14.7 $\Rightarrow$ $\mathcal{A}_0$ and $\mathcal{B}_0$ are isomorphic

$\Rightarrow$ Th($\mathcal{A}_0$) =Th($\mathcal{B}_0$)

$\Rightarrow$ Th($\mathcal{A}$) =Th($\mathcal{B}$) $\square$

Recall that $\mathbb{R}$ is **complete**:

for any subsets $A, B \subseteq \mathbb{R}$ with A'$<$'B
(i.e. $a < b$ for any $a \in A, b \in B$)
there is $\gamma \in \mathbb{R}$ with A'$\leq$'$\{\gamma\}$'$\leq$'B.

$\mathbb{Q}$ is *not* complete:

$$\text{take} \quad \begin{aligned} A &= \{x \in \mathbb{Q} \mid x < \pi\} \\ B &= \{x \in \mathbb{Q} \mid \pi < x\} \end{aligned}$$

14.9 Corollary
Th($\langle\mathbb{Q};<\rangle$) =Th($\langle\mathbb{R};<\rangle$)

In particular, the completeness of $\mathbb{R}$ is not a first-order property,

i.e. there is no $\Delta \subseteq$ Sent($\mathcal{L}$) such that for all $\mathcal{L}$-structures $\langle A;<\rangle$,

$$\langle A;<\rangle \models \Delta \text{ iff } \langle A;<\rangle \text{ is complete.}$$