15. Normal Forms

(a) Prenex Normal Form

A formula is in **prenex normal form (PNF)** if it has the form

$$Q_1 x_{i_1} Q_2 x_{i_2} \cdots Q_r x_{i_r}\, \psi,$$

where each Q_i is a quantifier
(i.e. either $\forall$ or $\exists$), and where
ψ is a formula containing no quantifiers.

15.1 PNF-Theorem
Every $\phi \in$ Form($\mathcal{L}$) is logically equivalent to an $\mathcal{L}$-formula in **PNF**.

Proof: Induction on ϕ
(working in the language with $\forall, \exists, \neg, \wedge$):

ϕ atomic: OK

$\phi = \neg\psi$,
say $\phi \leftrightarrow \neg Q_1 x_{i_1} Q_2 x_{i_2} \cdots Q_r x_{i_r}\, \chi$

Then $\phi \leftrightarrow Q_1^- x_{i_1} Q_2^- x_{i_2} \cdots Q_r^- x_{i_r}\, \neg\chi$,
where $Q^- = \exists$ if $Q = \forall$, and $Q^- = \forall$ if $Q = \exists$

$\phi = (\chi \wedge \rho)$ with χ, ρ in PNF
Note that $\vdash (\forall x_j \psi[x_j/x_i] \leftrightarrow \forall x_i \psi)$,
provided x_j does not occur in ψ (Ex. 12.5)

So w.l.o.g. the variables quantified over in χ do not occur in ρ and vice versa.

But then, e.g. $(\forall x \alpha \wedge \exists y \beta) \leftrightarrow \forall x \exists y (\alpha \wedge \beta)$ etc.
$\square$

(b) Skolem Normal Form

Recall: In the proof of CT, we introduced witnessing new constants for existential formulas such that

$\exists x \phi(x)$ is satisfiable iff $\phi(c)$ is satisfiable.

This way an $\exists x$ in front of a formula could be removed at the expense of a new constant.

Now we remove existential quantifiers 'inside' a formula at the expense of extra function symbols:

15.2 Observation:
Let $\phi = \phi(x, y)$ be an $\mathcal{L}$-formula with $x, y \in$Free(ϕ). Let f be a new unary function symbol (not in $\mathcal{L}$).

Then $\forall x \exists y \phi(x, y)$ *is satisfiable iff* $\forall x \phi(x, f(x))$ *is satisfiable.*
(f *is called a* **Skolem function** *for* ϕ*.)*

Proof: '$\Leftarrow$': clear

'$\Rightarrow$': Let $\mathcal{A}$ be an $\mathcal{L}$-structure with $\mathcal{A} \models \forall x \exists y \phi(x, y)$

$\Rightarrow$ for every $a \in A$ there is some $b \in A$ with $\phi(a, b)$

Interpret f by a function assigning to each $a \in A$ one such b
(this uses the Axiom of Choice!). $\square$

Example: $\mathbb{R} \models \forall x \exists y (x \doteq y^2 \vee x \doteq -y^2)$ – here $f(x) = \sqrt{| x |}$ will do.

15.3 Theorem

For every $\mathcal{L}$-formula ϕ
there is a formula $\phi^\star$
(with new constant and function symbols)
having only universal quantifiers in its PNF such that

$$\phi \text{ is satisfiable iff } \phi^\star \text{ is.}$$

More precisely,
any $\mathcal{L}$-structure $\mathcal{A}$
can be made into a structure $\mathcal{A}^\star$
interpreting the new constant and function symbols
such that

$$\mathcal{A} \models \phi \text{ iff } \mathcal{A}^\star \models \phi^\star.$$

16. Towards an uncountable language *(non-examinable)*

Our language $\mathcal{L}^{FOPC}$ is countable.
Now consider a first-order language of any cardinality $\kappa \geq \aleph_0$ by using

- k-ary predicate symbols $(P_\alpha^{(k)})_{\alpha<\kappa}$
- k-ary function symbols $(f_\beta^k)_{\beta<\kappa}$
- constant symbols $(c_\gamma)_{\gamma<\kappa}$

$\rightsquigarrow$ still get DT, CT and Compactness Theorem (using a bit more set theory)

16.1 Theorem: Löwenheim-Skolem $\uparrow$
Let $\mathcal{L}$ be a first-order language of cardinality $\kappa \geq \aleph_0$, let $\lambda \geq \kappa$ be any cardinal, and let $\mathcal{A}$ be an infinite $\mathcal{L}$-structure. Then there is an $\mathcal{L}$-structure $\mathcal{B} \equiv \mathcal{A}$ of cardinality λ.

Proof: Introduce new constant symbols $(d_\delta)_{\delta<\lambda}$ and let $\mathcal{B}_0$ be the term model of

$$\mathsf{Th}_{\mathcal{L}}(\mathcal{A}) \cup \{\neg d_\delta \doteq d_{\delta'} \mid \delta < \delta' < \lambda\}.$$

$\Rightarrow \mathcal{B}_0$ has cardinality λ
Let $\mathcal{B}$ be the $\mathcal{L}$-reduct of $\mathcal{B}_0$ (i.e., forget about the d_δ's) $\Rightarrow \mathcal{B} \equiv \mathcal{A}$ $\square$

Notation: p prime or $p = 0$, $\mathcal{L}_{ring} := \{+, \cdot; 0.1\}$
ACF$_p$:= the $\mathcal{L}_{ring}$-theory of algebraically closed fields of characteristic p.

16.2 Fact:
$K_1, K_2 \models$ **ACF**$_p$ s.t. $\sharp K_1 = \sharp K_2 > \aleph_0 \Rightarrow K_1 \simeq K_2$

16.3 Corollary:
ACF$_p$ *is maximally consisitent.*
Proof: Let $F_1, F_2 \models$ **ACF**$_p$
to show: $F_1 \equiv F_2$ (i.e., Th(F_1) =Th(F_2))
by L-S ($\downarrow$ and $\uparrow$), we find
$K_1 \equiv F_1$, $K_2 \equiv F_2$ s.t. $\sharp K_1 = \sharp K_2 = \sharp\mathbb{C}$
$\Rightarrow$ (16.2) $K_1 \simeq K_2$, so $F_1 \equiv K_1 \equiv K_2 \equiv F_2$ $\square$

16.4 Fact:
(a) *p prime, $n \in \mathbb{N} \Rightarrow$ there is a unique field $\mathbb{F}_{p^n}$ with $\sharp\mathbb{F}_{p^n} = p^n$ (unique up to isomorphism).*
(b) *If $m \mid n$ then $\mathbb{F}_{p^m} \subseteq \mathbb{F}_{p^n}$.*
(c) $\bar{\mathbb{F}}_p := \bigcup_{n\in\mathbb{N}} \mathbb{F}_{p^{n!}} \models$ **ACF**$_p$

16.5 Corollary to CT: *For any $\phi \in$Sent$(\mathcal{L}_{ring})$, **ACF**$_0 \models \phi$ iff **ACF**$_p \models \phi$ for almost all $p > 0$.*

16.6 Theorem: *Any injective polynomial function $f : \mathbb{C}^n \to \mathbb{C}^n$ is surjective.*

Proof: Let $\phi_{n,d} \in$ Sent$(\mathcal{L}_{ring})$ say this for all $f = (f_1, \ldots, f_n)$ of degree $\leq d$.

Then $\bar{\mathbb{F}}_p \models \phi_{n,d}$ for all primes p:
$f : \bar{\mathbb{F}}_p^n \to \bar{\mathbb{F}}_p^n$ polynomial of degree $\leq d$
and $\mathbf{y} = (y_1, \ldots, y_n) \in \bar{\mathbb{F}}_p^n$
$\Rightarrow \exists m$ s.t. $\mathbb{F}_{p^m}$ contains all y_i
and all coefficients of all f_j
$\Rightarrow \exists \mathbf{x} \in \mathbb{F}_{p^m}^n \subseteq \bar{\mathbb{F}}_p^n$ with $f(\mathbf{x}) = \mathbf{y}$

$\Rightarrow$ by Corollary 16.5, **ACF**$_0 \models \phi_{n,d}$
$\Rightarrow \mathbb{C} \models \phi_{n,d}$ $\square$