6.6 The Deduction Theorem for L_0

For any $\Gamma \subseteq$ *Form*$(\mathcal{L}_0)$ *and*
for any $\alpha, \beta \in$ *Form*$(\mathcal{L}_0)$*:*

if $\Gamma \cup \{\alpha\} \vdash \beta$ *then* $\Gamma \vdash (\alpha \to \beta)$.

Proof:

We prove by induction on m:

if $\alpha_1, \ldots, \alpha_m$ *is derivable in* L_0
from the hypotheses $\Gamma \cup \{\alpha\}$
then *for all* $i \leq m$
$(\alpha \to \alpha_i)$ *is derivable in* L_0
from the hypotheses Γ.

m=1

Either α_1 is an Axiom or $\alpha_1 \in \Gamma \cup \{\alpha\}$.

Case 1: α_1 is an Axiom
Then

1	α_1	[Axiom]
2	$(\alpha_1 \to (\alpha \to \alpha_1))$	[Instance of A1]
3	$(\alpha \to \alpha_1)$	[MP 1,2]

is a derivation of $(\alpha \to \alpha_1)$ from hypotheses $\emptyset$.

Note that if $\Delta \vdash \psi$ and $\Delta \subseteq \Delta'$, then obviously $\Delta' \vdash \psi$.

Thus $(\alpha \to \alpha_1)$ is derivable in L_0 from hypotheses Γ.

Case 2: $\alpha_1 \in \Gamma \cup \{\alpha\}$
If $\alpha_1 \in \Gamma$ then same proof as above works (with justification on line 1 changed to '$\in \Gamma$').

If $\alpha_1 = \alpha$, then, by Example 6.3, $\vdash (\alpha \to \alpha_1)$, hence $\Gamma \vdash (\alpha \to \alpha_1)$.

Induction Step

IH: Suppose result is true for derivations of length $\leq m$.

Let $\alpha_1, \ldots, \alpha_{m+1}$ be a derivation in L_0 from $\Gamma \cup \{\alpha\}$.

Then **either** α_{m+1} is an axiom
or $\alpha_{m+1} \in \Gamma \cup \{\alpha\}$ – in these cases proceed as above, even without IH.

Or α_{m+1} is obtained by MP from some earlier α_j, α_k, i.e. there are $j, k < m + 1$ such that $\alpha_j = (\alpha_k \to \alpha_{m+1})$.

By IH, we have

$$\begin{array}{ll} & \Gamma \vdash (\alpha \to \alpha_k) \\ \text{and} & \Gamma \vdash (\alpha \to \alpha_j), \\ \text{so} & \Gamma \vdash (\alpha \to (\alpha_k \to \alpha_{m+1})) \end{array}$$

Let $\beta_1, \ldots, \beta_r$ be a derivation in L_0 of $(\alpha \to \alpha_k) = \beta_r$ from Γ

and let $\gamma_1, \ldots, \gamma_s$ be a derivation in L_0 of $(\alpha \to (\alpha_k \to \alpha_{m+1})) = \gamma_s$ from Γ.

Then

1	β_1	
⋮	⋮	
r-1	β_{r-1}	
r	$(\alpha \to \alpha_k)$	
r+1	γ_1	
⋮	⋮	
r+s-1	γ_{s-1}	
r+s	$(\alpha \to (\alpha_k \to \alpha_{m+1}))$	
r+s+1	$((\alpha \to (\alpha_k \to \alpha_{m+1})) \to$ $((\alpha \to \alpha_k) \to (\alpha \to \alpha_{m+1})))$	[A2]
r+s+2	$((\alpha \to \alpha_k) \to (\alpha \to \alpha_{m+1}))$	[MP r+s, r+s+1]
r+s+3	$(\alpha \to \alpha_{m+1})$	[MP r, r+s+2]

is a derivation of $(\alpha \to \alpha_{m+1})$ in L_0 from Γ. □

6.7 Remarks

- Only needed instances of A1, A2 and the rule MP.
  So any system that includes A1, A2 and MP satisfies the Deduction Theorem.

- Proof gives a precise **algorithm** for converting any derivation showing $\Gamma \cup \{\alpha\} \vdash \beta$ into one showing $\Gamma \vdash (\alpha \to \beta)$.

- Converse is easy:

  If $\Gamma \vdash (\alpha \to \beta)$ then $\Gamma \cup \{\alpha\} \vdash \beta$.

  Proof:

⋮	⋮	derivation from Γ
r	$\alpha \to \beta$	
r+1	α	[$\in \Gamma \cup \{\alpha\}$]
r+2	β	[MP r, r+1]

□

6.8 Example of use of DT

If $\Gamma \vdash (\alpha \to \beta)$ and $\Gamma \vdash (\beta \to \gamma)$
then $\Gamma \vdash (\alpha \to \gamma)$.

Proof:

By the deduction theorem ('DT'), it suffices to show that $\Gamma \cup \{\alpha\} \vdash \gamma$.

⋮	⋮	proof from Γ
r	$(\alpha \to \beta)$	
r+1	⋮	
⋮	⋮	proof from Γ
r+s	$(\beta \to \gamma)$	
r+s+1	α	$[\in \Gamma \cup \{\alpha\}]$
r+s+2	β	[MP r, r+s+1]
r+s+3	γ	[MP r+s, r+s+2]

□

From now on we may treat DT as an additional inference rule in L_0.

6.9 Definition

The **sequent calculus** SQ is the system where a **proof** (or **derivation**) of $\phi \in \text{Form}(\mathcal{L}_0)$ from $\Gamma \subseteq \text{Form}(\mathcal{L}_0)$ is a finite sequence of **sequents**, i.e. of expressions of the form

$$\Delta \vdash_{SQ} \psi$$

with $\Delta \subseteq \text{Form}(\mathcal{L}_0)$ and $\Gamma \vdash_{SQ} \phi$ as last sequent.

Sequents may be formed according to the following rules

Ass: if $\psi \in \Delta$ then infer $\Delta \vdash_{SQ} \psi$

MP: from $\Delta \vdash_{SQ} \psi$ and $\Delta' \vdash_{SQ} (\psi \rightarrow \chi)$
infer $\Delta \cup \Delta' \vdash_{SQ} \chi$

DT: from $\Delta \cup \{\psi\} \vdash_{SQ} \chi$ infer $\Delta \vdash_{SQ} (\psi \rightarrow \chi)$

PC: from $\Delta \cup \{\neg\psi\} \vdash_{SQ} \chi$ and
$\Delta' \cup \{\neg\psi\} \vdash_{SQ} \neg\chi$ infer $\Delta \cup \Delta' \vdash_{SQ} \psi$

'PC' stands for *proof by contradiction*'

Note: no axioms.

6.10 Example of a proof in SQ

1	$\neg\beta \vdash_{SQ} \neg\beta$	[Ass]
2	$(\neg\beta \to \neg\alpha) \vdash_{SQ} (\neg\beta \to \neg\alpha)$	[Ass]
3	$(\neg\beta \to \neg\alpha), \neg\beta \vdash_{SQ} \neg\alpha$	[MP 1,2]
4	$\alpha, \neg\beta \vdash_{SQ} \alpha$	[Ass]
5	$(\neg\beta \to \neg\alpha), \alpha \vdash_{SQ} \beta$	[PC 3,4]
6	$(\neg\beta \to \neg\alpha) \vdash_{SQ} (\alpha \to \beta)$	[DT 5]
7	$\vdash_{SQ} ((\neg\beta \to \neg\alpha) \to (\alpha \to \beta))$	[DT 6]

So $\vdash_{SQ}$ A3.

We'd better write '$\Gamma \vdash_{L_0} \phi$' for '$\Gamma \vdash \phi$ in L_0'.

6.11 Theorem

L_0 and SQ are equivalent: for all Γ, ϕ

$$\Gamma \vdash_{L_0} \phi \textit{ iff } \Gamma \vdash_{SQ} \phi.$$

Proof: Exercise