7. Consistency, Completeness and Compactness

7.1 Definition

Let $\Gamma \subseteq \text{Form}(\mathcal{L}_0)$. Γ is said to be **consistent** (or $\mathcal{L}_0$-consistent) if for *no* formula α both $\Gamma \vdash \alpha$ and $\Gamma \vdash \neg\alpha$.

Otherwise Γ is **inconsistent**.

E.g. $\emptyset$ is consistent: by soundness theorem, α and $\neg\alpha$ are never simultaneously true.

7.2. Lemma

$\Gamma \cup \{\neg\phi\}$ is inconsistent iff $\Gamma \vdash \phi$.
(In part., if $\Gamma \nvdash \phi$ then $\Gamma \cup \{\neg\phi\}$ is consistent).

Proof: **'$\Leftarrow$':**

$$\Gamma \vdash \phi \Rightarrow \left.\begin{array}{lcl} \Gamma \cup \{\neg\phi\} & \vdash & \phi \\ \Gamma \cup \{\neg\phi\} & \vdash & \neg\phi \end{array}\right\} \Rightarrow \begin{array}{l} \Gamma \cup \{\neg\phi\} \\ \text{is inconsistent} \end{array}$$

'$\Rightarrow$':

$$\left.\begin{array}{lcl} \Gamma \cup \{\neg\phi\} & \vdash & \alpha \\ \Gamma \cup \{\neg\phi\} & \vdash & \neg\alpha \end{array}\right\} \Rightarrow_{6.11} \left.\begin{array}{lcl} \Gamma \cup \{\neg\phi\} & \vdash_{SQ} & \alpha \\ \Gamma \cup \{\neg\phi\} & \vdash_{SQ} & \neg\alpha \end{array}\right\}$$

$$\Rightarrow_{\text{PC}} \quad \Gamma \vdash_{SQ} \phi \quad \Rightarrow_{6.11} \quad \Gamma \vdash \phi$$

$\square$

7.3 Lemma
Suppose Γ *is consistent and* $\Gamma \vdash \phi$*.*
Then $\Gamma \cup \{\phi\}$ *is consistent.*

Proof: Suppose not, i.e. for some α

$$\left.\begin{array}{l}\Gamma \cup \{\phi\} \vdash \alpha \\ \Gamma \cup \{\phi\} \vdash \neg\alpha\end{array}\right\} \Rightarrow_{\mathrm{DT}} \left.\begin{array}{l}\Gamma \vdash (\phi \to \alpha) \\ \Gamma \vdash (\phi \to \neg\alpha)\end{array}\right\} \overset{\Gamma \vdash \phi}{\Rightarrow}_{\mathrm{MP}}$$

$$\Rightarrow \begin{array}{l}\Gamma \vdash \alpha \\ \Gamma \vdash \neg\alpha\end{array} \quad ⨳$$

□

7.4 Definition
$\Gamma \subseteq \mathrm{Form}(\mathcal{L}_0)$ is **maximal consistent** if
(i) Γ is consistent, and
(ii) for *every* ϕ, either $\Gamma \vdash \phi$ or $\Gamma \vdash \neg\phi$.

Note: This is equivalent to saying that for every ϕ, if $\Gamma \cup \{\phi\}$ is consistent then $\Gamma \vdash \phi$.
Proof: Exercise

7.5 Lemma

Suppose Γ *is maximal consistent.*
Then for every $\psi, \chi \in$ *Form*$(\mathcal{L}_0)$
(a) $\Gamma \vdash \neg\psi$ *iff* $\Gamma \nvdash \psi$
(b) $\Gamma \vdash (\psi \to \chi)$ *iff either* $\Gamma \vdash \neg\psi$ *or* $\Gamma \vdash \chi$.

Proof:

(a) **'$\Rightarrow$':** by consistency
'$\Leftarrow$': by maximality

(b) **'$\Rightarrow$':** Suppose $\Gamma \nvdash \neg\psi$ and $\Gamma \nvdash \chi$
$\Rightarrow$ $\Gamma \vdash \psi$ and $\Gamma \vdash \neg\chi$
$\Gamma \vdash (\psi \to \chi) \Rightarrow_{\mathrm{MP}} \Gamma \vdash \chi$ ⨳

'$\Leftarrow$': Suppose $\Gamma \vdash \neg\psi$
$\Gamma \vdash (\neg\psi \to (\psi \to \chi))$ - Problems $\sharp$ 2, (5)(i)
$\Rightarrow_{\mathrm{MP}} \Gamma \vdash (\psi \to \chi)$

Suppose $\Gamma \vdash \chi$
$\Gamma \vdash (\chi \to (\psi \to \chi))$ - Axiom A1
$\Rightarrow_{\mathrm{MP}} \Gamma \vdash (\psi \to \chi)$

$\square$

7.6 Theorem

Suppose Γ is maximal consistent.
Then Γ is satisfiable.

Proof:
For each i, $\Gamma \vdash p_i$ or $\Gamma \vdash \neg p_i$ (by maximality), but not both (by consistency)

Define a valuation v by

$$v(p_i) = \begin{cases} T & \text{if } \Gamma \vdash p_i \\ F & \text{if } \Gamma \vdash \neg p_i \end{cases}$$

Claim: for all $\phi \in \text{Form}(\mathcal{L}_0)$:

$$\widetilde{v}(\phi) = T \text{ iff } \Gamma \vdash \phi$$

Proof by induction on the length n of ϕ:

n=1:
Then $\phi = p_i$ for some i, and so, by def. of v,

$$\widetilde{v}(p_i) = T \text{ iff } \Gamma \vdash p_i.$$

IH: Claim true for all $i \leq n$.

Now assume length (ϕ) = n+1

Case 1: $\phi = \neg\psi$ ($\Rightarrow$ length (ψ) = n)

$\widetilde{v}(\phi) = T$	iff	$\widetilde{v}(\psi) = F$	tt $\neg$
	iff	$\Gamma \nvdash \psi$	IH
	iff	$\Gamma \vdash \neg\psi$	7.5(a)
	iff	$\Gamma \vdash \phi$	

Case 2: $\phi = (\psi \rightarrow \chi)$
($\Rightarrow$ length (ψ), length (χ) $\leq$ n)

$\widetilde{v}(\phi) = T$	iff	$\widetilde{v}(\psi) = F$ or $\widetilde{v}(\chi) = T$	tt $\rightarrow$
	iff	$\Gamma \nvdash \psi$ or $\Gamma \vdash \chi$	IH
	iff	$\Gamma \vdash \neg\psi$ or $\Gamma \vdash \chi$	7.5(a)
	iff	$\Gamma \vdash (\psi \rightarrow \chi)$	7.5(b)
	iff	$\Gamma \vdash \phi$	

So $\widetilde{v}(\phi) = T$ for all $\phi \in \Gamma$, i.e. v satisfies Γ.

$\square$

7.7 Theorem
Suppose Γ is consistent. Then there is a maximal consistent Γ' such that $\Gamma \subseteq \Gamma'$.

Proof:

Form($\mathcal{L}_0$) is countable, say

$$\text{Form}(\mathcal{L}_0) = \{\phi_1, \phi_2, \phi_3, \ldots\}.$$

Construct consistent sets

$$\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \ldots$$

as follows: $\Gamma_0 := \Gamma$.

Having constructed Γ_n consistently, let

$$\Gamma_{n+1} := \begin{cases} \Gamma_n \cup \{\phi_{n+1}\} & \text{if} \;\; \Gamma_n \vdash \phi_{n+1} \\ \Gamma_n \cup \{\neg\phi_{n+1}\} & \text{if} \;\; \Gamma_n \nvdash \phi_{n+1} \end{cases}$$

Then Γ_{n+1} is consistent by 7.3 and 7.2.

Now let $\Gamma' := \bigcup_{n=0}^{\infty} \Gamma_n$.

Then Γ' is consistent:

Any proof of $\Gamma' \vdash \alpha$ and $\Gamma' \vdash \neg\alpha$ would use only finitely many formulas from Γ', so for some n, $\Gamma_n \vdash \alpha$ and $\Gamma_n \vdash \neg\alpha$ – contradicting the consistency of Γ_n.

Finally, Γ' is maximal (even in a stronger sense): for all n, either $\phi_n \in \Gamma'$ or $\neg\phi_n \in \Gamma'$. $\square$

Note that the proof does not make use of Zorn's Lemma.

7.8 Corollary
If Γ is consistent then Γ is satisfiable.

Proof: 7.6 + 7.7 $\square$

7.9 The Completeness Theorem

If $\Gamma \models \phi$ then $\Gamma \vdash \phi$.

Proof:

Suppose $\Gamma \models \phi$, but $\Gamma \nvdash \phi$.

$\Rightarrow$ by 7.2, $\Gamma \cup \{\neg\phi\}$ is consistent
$\Rightarrow$ by 7.8, there is some valuation v such that $\tilde{v}(\psi) = T$ for all $\psi \in \Gamma \cup \{\neg\phi\}$
$\Rightarrow$ $\tilde{v}(\psi) = T$ for all $\psi \in \Gamma$, but $\tilde{v}(\phi) = F$
$\Rightarrow$ $\Gamma \not\models \phi$: contradiction. $\square$

7.10 Corollary

(7.9 Completeness + 6.5 Soundness)

$$\Gamma \models \phi \text{ iff } \Gamma \vdash \phi$$

7.11 The Compactness Theorem for L_0

$\Gamma \subseteq Form(\mathcal{L}_0)$ is satisfiable iff every finite subset of Γ is satisfiable.

Proof: **'$\Rightarrow$':** obvious –
if $\tilde{v}(\psi) = T$ for all $\psi \in \Gamma$ then $\tilde{v}(\psi) = T$ for all $\psi \in \Gamma' \subseteq \Gamma$.

'$\Leftarrow$':
Suppose every finite $\Gamma' \subseteq \Gamma$ is satisfiable, but Γ is not.

Then, by 7.8, Γ is inconsistent, i.e. $\Gamma \vdash \alpha$ and $\Gamma \vdash \neg\alpha$ for some α.

But then, for some *finite* $\Gamma' \subseteq \Gamma$:

$$\begin{aligned} & \Gamma' \vdash \alpha \text{ and } \Gamma' \vdash \neg\alpha \\ \Rightarrow\ & \Gamma' \models \alpha \text{ and } \Gamma' \models \neg\alpha \text{ (by soundness)} \\ \Rightarrow\ & \Gamma' \text{ not satisfiable: contradiction.} \end{aligned}$$

□