

PART II:

PREDICATE CALCULUS

so far:

- *logic of the connectives* $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \dots$ (as used in mathematics)
- *smallest unit*: propositions
- *deductive calculus*: checking logical validity and computing truth tables
- $>$ sound, complete, compact

now:

- look *more deeply into* the structure of propositions used in mathematics
- analyse grammatically correct use of *functions, relations, constants, variables* and *quantifiers*
- define *logical validity* in this refined language
- discover *axioms* and *rules of inference* (beyond those of propositional calculus) used in mathematical arguments
- prove: — $>$ sound, complete, compact

8. The language of (first-order) predicate calculus

The language \mathcal{L}^{FOPC} consists of the following symbols:

Logical symbols

connectives: \rightarrow, \neg

quantifier: \forall ('for all')

variables: x_0, x_1, x_2, \dots

3 punctuation marks: $() ,$

equality symbol: \doteq

non-logical symbols:

predicate (or relation) symbols: $P_n^{(k)}$ for $n \geq 0, k \geq 1$ ($P_n^{(k)}$ is a k -ary predicate symbol)

function symbols: $f_n^{(k)}$ for $n \geq 0, k \geq 1$ ($f_n^{(k)}$ is a k -ary function symbol)

constant symbols: c_n for $n \geq 0$

8.1 Definition

(a) The **terms** of \mathcal{L}^{FOPC} are defined recursively as follows:

(i) Every variable is a term.

(ii) Every constant symbol is a term.

(iii) For each $n \geq 0, k \geq 1$, if t_1, \dots, t_k are terms, so is the string

$$f_n^{(k)}(t_1, \dots, t_k)$$

(b) An **atomic formula** of \mathcal{L}^{FOPC} is any string of the form

$$P_n^{(k)}(t_1, \dots, t_k) \text{ or } t_1 \doteq t_2$$

with $n \geq 0, k \geq 1$, and where all t_i are terms.

(c) The **formulas** of \mathcal{L}^{FOPC} are defined recursively as follows:

(i) Any atomic formula is a formula

(ii) If ϕ, ψ are formulas, then so are $\neg\phi$ and $(\phi \rightarrow \psi)$

(iii) If ϕ is a formula, then for any variable x_i so is $\forall x_i \phi$

8.2 Examples

c_0 ; c_3 ; x_5 ; $f_3^{(1)}(c_2)$; $f_4^{(2)}(x_1, f_3^{(1)}(c_2))$ are all terms

$f_2^{(3)}(x_1, x_2)$ is *not* a term (wrong arity)

$P_0^{(3)}(x_4, c_0, f_3^{(2)}(c_1, x_2))$ and $f_1^{(2)}(c_5, c_6) \doteq x_{11}$ are atomic formulas

$f_3^{(1)}(c_2)$ is a term, but no formula

$\forall x_1 f_2^{(2)}(x_1, c_7) \doteq x_2$ is a formula, not atomic

$\forall x_2 P_0^{(1)}(x_3)$ is a formula

8.3 Remark

We have **unique readability** for terms, for atomic formulas, and for formulas.

8.4 Interpretations and logical validity for \mathcal{L}^{FOPC} (Informal discussion)

(A) Consider the formula

$$\phi_1 : \forall x_1 \forall x_2 (x_1 \doteq x_2 \rightarrow f_5^{(1)}(x_1) \doteq f_5^{(1)}(x_2))$$

Given that \doteq is to be interpreted as equality, \forall as ‘for all’, and the $f_n^{(k)}$ as actual functions (in k arguments), ϕ_1 *should always be true*. We shall write

$$\models \phi_1$$

and say ‘ ϕ_1 is **logically valid**’.

(B) Consider the formula

$$\phi_2 : \forall x_1 \forall x_2 (f_7^{(2)}(x_1, x_2) \doteq f_7^{(2)}(x_2, x_1) \rightarrow x_1 \doteq x_2)$$

Then ϕ_2 may be false or true depending on the situation:

- If we interpret $f_7^{(2)}$ as $+$ on \mathbf{N} , ϕ_2 becomes false, e.g. $1+2=2+1$, but $1 \neq 2$. So in this interpretation, ϕ_2 is false and $\neg\phi_2$ is true. Write

$$\langle \mathbf{N}, + \rangle \models \neg\phi_2$$

- If we interpret $f_7^{(2)}$ as $-$ on \mathbf{R} , ϕ_2 becomes true: if $x_1 - x_2 = x_2 - x_1$, then $2x_1 = 2x_2$, and hence $x_1 = x_2$.

So

$$\langle \mathbf{R}, - \rangle \models \phi_2$$

8.5 Free and bound variables

(Informal discussion)

There is a further complication: Consider the formula

$$\phi_3 : \forall x_0 P_0^{(2)}(x_1, x_0)$$

Under the interpretation $\langle \mathbf{N}, \leq \rangle$ you cannot tell whether $\langle \mathbf{N}, \leq \rangle \models \phi_3$:

- if we put $x_1 = 0$ then yes
- if we put $x_1 = 2$ then no.

So it depends on the value we assign to x_1 (like in propositional calculus: truth value of $p_0 \wedge p_1$ depends on the valuation).

In ϕ_3 we *can* assign a value to x_1 because x_1 occurs **free** in ϕ_3 .

For x_0 , however, it makes no sense to assign a particular value; because x_0 is **bound** in ϕ_3 by the quantifier $\forall x_0$.