2. Valuations

Propositional Calculus

- is designed to find the **truth** or **falsity** of a compound formula from its constituent parts

- it computes the **truth values** T (‘true’) or F (‘false’) of a formula ϕ, given the truth values assigned to the smallest constituent parts, i.e. the propositional variables occuring in ϕ

How this can be done is made precise in the following definition.

2.1 Definition

1. A **valuation** v is a function

$$v : \{p_0, p_1, p_2, \ldots\} \to \{T, F\}$$

2. Given a valuation v we extend v uniquely to a function

$$\widetilde{v} : \text{Form } (\mathcal{L}) \to \{T, F\}$$

(Form ($\mathcal{L}$) denotes the set of all formulas of $\mathcal{L}$)

defined recursively as follows:
2.(i) If ϕ is a formula of length 1, i.e. a propositional variable, then $\widetilde{v}(\phi) := v(\phi)$.
2.(ii) If $\widetilde{v}$ is defined for all formulas of length $\leq n$, let ϕ be a formula of length $n+1$ (≥ 2).

Then, by the Unique Readability Theorem,
either $\phi = \neg\psi$ for a unique ψ
or $\phi = (\psi \star \chi)$ for a unique pair ψ, χ and a unique $\star \in \{\to, \wedge, \vee, \leftrightarrow\}$,

where ψ and χ are formulas of lenght $\leq n$, so $\widetilde{v}(\psi)$ and $\widetilde{v}(\chi)$ are already defined.

Truth Tables

Define $\widetilde{v}(\phi)$ by the following truth tables:

Negation

ψ	$\neg\psi$
T	F
F	T

i.e. if $\widetilde{v}(\psi) = T$ then $\widetilde{v}(\neg\psi) = F$
and if $\widetilde{v}(\psi) = F$ then $\widetilde{v}(\neg\psi) = T$

Binary Connectives

ψ	χ	$\psi \to \chi$	$\psi \wedge \chi$	$\psi \vee \chi$	$\psi \leftrightarrow \chi$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	T	F	T	F
F	F	T	F	F	T

so, e.g., if $\widetilde{v}(\psi) = F$ and $\widetilde{v}(\chi) = T$
then $\widetilde{v}(\psi \vee \chi) = T$ etc.

Remark: These truth tables correspond roughly to our ordinary use of the words *'not'*, *'if - then'*, *'and'*, *'or'* and *'if and only if'*, except, perhaps, the truth table for implication ($\rightarrow$).

2.2 Example

Construct the full truth table for the formula

$$\phi := ((p_0 \vee p_1) \rightarrow \neg(p_1 \wedge p_2))$$

$\tilde{v}(\phi)$ only depends on $v(p_0), v(p_1)$ and $v(p_2)$.

p_o	p_1	p_2	$(p_0 \vee p_1)$	$(p_1 \wedge p_2)$	$\neg(p_1 \wedge p_2)$	ϕ
T	T	T	T	T	F	F
T	T	F	T	F	T	T
T	F	T	T	F	T	T
T	F	F	T	F	T	T
F	T	T	T	T	F	F
F	T	F	T	F	T	T
F	F	T	F	F	T	T
F	F	F	F	F	T	T

2.3 Example Truth table for

$$\phi := ((p_0 \to p_1) \to (\neg p_1 \to \neg p_0))$$

p_0	p_1	$(p_0 \to p_1)$	$\neg p_1$	$\neg p_0$	$(\neg p_1 \to \neg p_0)$	ϕ
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

3. Logical Validity

3.1 Definition

- A valuation v **satisfies** a formula ϕ if $\tilde{v}(\phi) = T$

- If a formula ϕ is satisfied by *every* valuation then ϕ is **logically valid** or a **tautology** (e.g. Example 2.3, not Example 2.2)
  Notation: $\models \phi$

- If a formula ϕ is satisfied by *some* valuation then ϕ is **satisfiable** (e.g. Example 2.2)

- A formula ϕ is a **logical consequence** of a formula ψ if, for *every* valuation v:

  $$\text{if } \tilde{v}(\psi) = T \text{ then } \tilde{v}(\phi) = T$$

  Notation: $\psi \models \phi$

3.2 Lemma $\psi \models \phi$ *if and only if* $\models (\psi \to \phi)$.

Proof: '$\Rightarrow$': Assume $\psi \models \phi$.
Let v be any valuation.

- If $\tilde{v}(\psi) = T$ then (by def.) $\tilde{v}(\phi) = T$, so $\tilde{v}((\psi \to \phi)) = T$ by tt $\to$.

('tt $\star$' stands for the truth table of the connective $\star$)

- If $\tilde{v}(\psi) = F$ then $\tilde{v}((\psi \to \phi)) = T$ by tt $\to$.

Thus, for every valuation v, $\tilde{v}((\psi \to \phi)) = T$, so $\models (\psi \to \phi)$.

'$\Leftarrow$': Conversely, suppose $\models (\psi \to \phi)$.
Let v be any valuation s.t. $\tilde{v}(\psi) = T$.
Since $\tilde{v}((\psi \to \phi)) = T$, also $\tilde{v}(\phi) = T$ by tt $\to$.
Hence $\psi \models \phi$.

$\square$

More generally, we make the following

3.3 Definition Let Γ be any (possibly infinite) set of formulas and let ϕ be any formula. Then ϕ is a **logical consequence** of Γ if, for every valuation v:

$$\text{if } \tilde{v}(\psi) = T \text{ for all } \psi \in \Gamma \text{ then } \tilde{v}(\phi) = T$$

Notation: $\Gamma \models \phi$

3.4 Lemma
$\Gamma \cup \{\psi\} \models \phi$ *if and only if* $\Gamma \models (\psi \to \phi)$.

Proof: similar to the proof of previous lemma 3.2 - Exercise.

3.5 Example

	$\models ((p_0 \to p_1) \to (\neg p_1 \to \neg p_0))$	(cf. Ex. 2.3
Hence	$(p_0 \to p_1) \models (\neg p_1 \to \neg p_0)$	by 3.2
Hence	$\{(p_0 \to p_1), \neg p_1\} \models \neg p_0$	by 3.4

3.6 Example

$$\phi \models (\psi \to \phi)$$

Proof:

If $\widetilde{v}(\phi) = T$ then, by tt $\to$, $\widetilde{v}((\psi \to \phi)) = T$ (no matter what $\widetilde{v}(\psi)$ is).

$\square$