1 The lack of memory property

For hand in times please consult MINERVA. Questions 7-10 are practice/revision questions.

1. Show that among continuous random variables, only the exponential distribution has the lack of memory property: let Z be any continuous random variable with $\mathbb{P}(Z > 0) = 1$, whose distribution has the lack of memory property, i.e. $\mathbb{P}(Z - t > s | Z > t) = \mathbb{P}(Z > s)$ for all $s \geq 0$, $t \geq 0$ (i.e. for all $s, t \in [0, \infty)$); by studying the function $G(t) := \mathbb{P}(Z > t)$, show that $Z \sim \text{Exp}(\mu)$ for some $\mu > 0$.

2. Let Z be an exponential random variable and R an independent nonnegative random variable. Show that Z has the **lack of memory property** also at the random time R, i.e.

$$\mathbb{P}(Z - R > u | Z > R) = \mathbb{P}(Z > u).$$

Hint: Calculate $\mathbb{P}(Z > R + u)$ by conditioning on Z.

3. Let $X_1, \ldots, X_n \sim \text{Exp}(\lambda)$ be independent exponential clocks. Show that for the first clock to ring, we have $W = \min\{X_1, \ldots, X_n\} \sim \text{Exp}(n\lambda)$. Using the lack of memory property, argue that the further time until the second clock rings is $\text{Exp}((n-1)\lambda)$ and independent of W.
Hint: Let $K = k$ if $W = X_k$.

4. *Let X and Y be independent exponential random variables (competing exponential alarm clocks) with respective parameters λ and μ. Let*

$$W = \min\{X, Y\},\ Z = \max\{X, Y\},\ O = Z - W,\ M = 1_{\{X \leq Y\}} = \begin{cases} 1 & \text{if } X \leq Y, \\ 0 & \text{if } X > Y. \end{cases}$$

   (a) *Calculate $\mathbb{P}(W > s)$ and $\mathbb{P}(M = 1)$. Identify the distributions of W and M. Show that the events $\{W > s\}$ and $\{M = 1\}$ are independent.*

   (b) *Express the event $\{W \leq w, M = 1, O \leq t\}$ in terms of X and Y and calculate its probability. What is $\mathbb{P}(W \leq w, M = 0, O \leq t)$?*

   Are W and (M, O) independent? You may assume without proof that it is enough to check that $\mathbb{P}(W \in A, (M, O) \in B) = \mathbb{P}(W \in A)\mathbb{P}((M, O) \in B)$ for $A = [0, w]$ and $B = \{m\} \times [0, t]$ for all $w \geq 0$, $m \in \{0, 1\}$ and $t \geq 0$.

   Are M and O independent? Find the conditional distributions of O given $M = 0$, and of O given $M = 1$. Is this related to the lack of memory property?

5. *A bank has two clerks. Service times at this bank are independent* $\text{Exp}(\mu)$. *When the bank opens at 9am, you enter the bank together with two other customers. You are generous and let the other two customers proceed to be served. You will then be served by the next available clerk; what is the probability that, of the three customers, you will be the last to leave?*

   Hint: Express the event in question in terms of three independent $\text{Exp}(\mu)$ random variables.

We write $\text{PP}(\lambda)$ *to refer to a Poisson process of rate λ. You have seen at least two different ways of characterising a* $\text{PP}(\lambda)$*, which are useful in different situations, as you will see here.*

6. **Superposition.** Suppose that $(X_t)_{t\geq 0} \sim \text{PP}(\lambda)$ and $(Y_t)_{t\geq 0} \sim \text{PP}(\mu)$ are independent. Let $Z_t = X_t + Y_t$, $t \geq 0$. Show that $(Z_t)_{t\geq 0} \sim \text{PP}(\lambda + \mu)$.

   Hint: it is not enough to check that $Z_t \sim \text{Poi}((\lambda + \mu)t)$. What more do we need?

7. **Thinning.** Suppose that a radioactive source emits particles at times of a PP(λ). Suppose that a Geiger counter detects each particle independently with probability $p \in (0,1)$. What is the distribution of the time until the first particle is detected? Deduce that the counting process of particles detected is a PP(λp).

   Hint: you may find it convenient to calculate the moment generating function of the time until the first particle is detected.

8. Find the **finite-dimensional distributions** of a PP(λ), i.e.

$$\mathbb{P}(X_{t_1} = x_1, \dots, X_{t_n} = x_n), \qquad 0 \le t_1 \le \dots \le t_n,\ 0 \le x_1 \le \dots \le x_n,\ n \ge 1.$$

9. **Linear time-change.** Suppose that $Z \sim \text{Exp}(1)$. Show that $Z/\lambda \sim \text{Exp}(\lambda)$. Deduce that if $(X_t)_{t\ge 0} \sim \text{PP}(1)$ and $Y_t = X_{\lambda t}$, then $(Y_t)_{t\ge 0} \sim \text{PP}(\lambda)$.

Questions 10-12 are optiional and won't be graded (and thus doesn't need to be handed in). They are meant to deepen your understanding of the earlier material and go a little beyond the scope of the course. There will probably not be time for them to be covered in the classes, but full solutions will be given on the solution sheets.

10. We are going to show that a right-continuous, integer-valued, increasing process $(X_t, t \ge 0)$ starting from 0 is a Poisson process with rate $\lambda > 0$ if and only if it has independent increments and, as $h \searrow 0$, uniformly in t

$$\mathbb{P}(X_{t+h} - X_t = 0) = 1 - \lambda h + o(h), \qquad \mathbb{P}(X_{t+h} - X_t = 1) = \lambda h + o(h).$$

   Here the first statement above means that

$$\frac{\mathbb{P}(X_{t+h} - X_t = 0) - 1 + \lambda h}{h} \to 0 \text{ as } h \to 0 \text{ uniformly in } t$$

   and the second means

$$\frac{\mathbb{P}(X_{t+h} - X_t = 1) - \lambda h}{h} \to 0 \text{ as } h \to 0 \text{ uniformly in } t.$$

   (a) Prove the "only if" direction, i.e. suppose that $X \sim$PP(λ) and show that as $h \searrow 0$, uniformly in t

$$\mathbb{P}(X_{t+h} - X_t = 0) = 1 - \lambda h + o(h), \qquad \mathbb{P}(X_{t+h} - X_t = 1) = \lambda h + o(h).$$

   (b) Let $p_j(t) = \mathbb{P}(X_t = j)$. Show that, uniformly in t

$$\frac{p_j(t+h) - p_j(t)}{h} = -\lambda p_j(t) + \lambda p_{j-1}(t) + O(h)$$

   where the meaning of the $O(h)$ notation is that there exists a constant C such that for all t

$$\left| \frac{p_j(t+h) - p_j(t)}{h} + \lambda p_j(t) - \lambda p_{j-1}(t) \right| \le Ch.$$

   (c) Show that the $p_j(t), j = 0, 1, \dots$ satisfies a system of ODE's which you should solve.

11. **Excess lifetime.** Let T_n be the time of the nth arrival in a Poisson process X with rate λ, and define the excess lifetime process $E = (E_t)_{t\ge 0}$ by

$$E_t = T_{X_t + 1} - t.$$

   This is the time one must wait after time t before the next arrival.

(a) Sketch a realisation of X illustrating E_t for a fixed $t > 0$. Show E in a second sketch underneath.

(b) Show by conditioning on T_1 that $\mathbb{P}(E_t > x) = e^{-\lambda(t+x)} + \int_0^t \mathbb{P}(E_{t-u} > x)\lambda e^{-\lambda u} du.$

(c) Solve this integral equation to find $\mathbb{P}(E_t > x)$, the so-called survival function of E_t.

(d) How does this relate to the *lack of memory property*?

12. **Inhomogeneous Poisson processes.** Suppose that we want to model a process where the rate of arrivals depends on how much time has elapsed (for example, this might be appropriate for the radioactive decay of certain substances). Let $\lambda\colon [0,\infty) \to [0,\infty)$ be a continuous function such that $\Lambda(t) := \int_0^t \lambda(s)ds < \infty$ and $\int_t^\infty \lambda(s)ds = \infty$ for all $t \geq 0$. We want a process $(Y_t)_{t\geq 0}$ such that the rate of jumping at time $t \geq 0$ is $\lambda(t)$.

    Let $(X_t)_{t\geq 0} \sim \mathrm{PP}(1)$ and define $Y_t = X_{\Lambda(t)}$, $t \geq 0$. The process $(Y_t)_{t\geq 0}$ is called an inhomogeneous Poisson process of rate $\lambda(t)$. Note that if $\lambda(t) \equiv \lambda$, then we just have an ordinary $\mathrm{PP}(\lambda)$, by Question 10.

    (a) What is the distribution of the number of jumps in an interval $(t, t+s]$?

    (b) Show that $\lim_{h\to 0} \frac{1}{h}\mathbb{P}(\text{there is at least one jump in } (t, t+h]) = \lambda(t)$, as desired.

    (c) Show that $(Y_t)_{t\geq 0}$ has independent increments.

    (d) Let $(Y_t)_{t\geq 0}$ have holding times $Z_0, Z_1, \ldots$ and jump times $T_1, T_2, \ldots$. Find the conditional density of Z_n given $T_n = t$. Deduce that, in general, $Z_0, Z_1, \ldots$ are dependent.