4 Renewal theory (Sheet4 for classes)

Questions 7 to 11 will not be marked. Please see MINERVA for hand in times. This sheet will be covered in the final class.

1. **Inspection paradox.** Suppose that buses arrive at a bus stop as a Poisson process of rate λ. Consider the duration D_t of the inter-arrival time containing t, i.e. $D_t = A_t + E_t$, where, at time t, E_t is the time until the next bus arrives, and A_t is the time since the last one has passed (and $A_t = t$ if no bus arrived in $[0, t]$). What is the distribution of E_t? What is the distribution of A_t? Show that $\mathbb{E}(D_t) > 1/\lambda$, so that the inter-arrival time we see ("inspect") has larger mean than a standard inter-arrival time.

2. **Waiting time paradox.** Consider the Gamma(a, λ) distribution with density

$$f(x) = \frac{\lambda^a}{\Gamma(a)} x^{a-1} e^{-\lambda x}, \qquad x > 0\,.$$

   (a) Calculate and identify the associated size-biased distribution.

   (b) Suppose that the counting process of buses at a particular bus stop can be modelled by a renewal process X with stationary increments and Gamma(a, λ) inter-arrival times. Calculate the average waiting time m_{stat} of a customer arriving at time t.

   (c) Also calculate the average waiting time m_{ren} of a customer arriving just after a bus has passed. Deduce that

$$m_{\text{stat}} > m_{\text{ren}} \iff a < 1$$

   This is a version of the waiting time paradox. What is paradoxical here?

3. Let X be an (undelayed) renewal process with finite mean inter-renewal times with density f. Let $m(t) = \mathbb{E}(X_t)$ be the associated renewal function. Recall that $m'(t) = \sum_{k=1}^{\infty} f^{*(k)}(t)$.

   (a) Suppose that $H\colon [0, \infty) \to \mathbb{R}$ is bounded on bounded intervals. Show that the function $r = H + H * m'$ satisfies the *renewal-type equation* $r = H + r * f$.

   (b) (For the keen) Show that $r = H + H * m'$ is, in fact, the unique solution.

   (c) Show that the distribution of the excess lifetime $E_t = T_{X_t+1} - t$ satisfies

$$\mathbb{P}(E_t > y) = \overline{F}(t + y) + \int_0^t \overline{F}(t + y - x) m'(x) dx, \qquad \text{where } \overline{F}(t) = \int_t^{\infty} f(s) ds.$$

   (d) Let X be a renewal process with continuous inter-renewal times of finite mean μ. Deduce, using the Key Renewal Theorem, that the limit of $\mathbb{P}(E_t > y)$ as $t \to \infty$ is

$$\frac{1}{\mu} \int_y^{\infty} \overline{F}(z) dz.$$

4. Let X be a renewal process with 1-arithmetic (in particular integer-valued) inter-renewal times Z_j of finite mean μ.

   (i) Show that $E_n = T_{X_n+1} - n$ is a discrete-time Markov chain.

   (ii) Calculate its stationary distribution and deduce that

$$\mathbb{P}(E_n = k) \to \mu^{-1} \mathbb{P}(Z_1 \geq k) \qquad \text{as } n \to \infty.$$

(iii) Show that $\mu^{-1}\mathbb{P}(Z_1 \geq k)$ is the probability function of a random variable U picked uniformly from $\{1, \dots, S\}$ conditionally given S, where S has the size-biased distribution associated with the distribution of Z_1.

5. A branch of an insurance company has at its disposal an initial capital of $u > 0$ at time $t = 0$ and receives linear premium income, ct by time $t \geq 0$, from which it has to meet claims A_n, $n \geq 1$, of independent exponential sizes with parameter μ, arriving at the times T_n, $n \geq 1$, of a Poisson process $(N_t)_{t\geq 0}$ with rate λ. Denote the reserve at time t by R_t.

   (a) Using $Z_i = R_{\varepsilon i} - R_{\varepsilon(i-1)}$, $i \geq 1$, or otherwise, show that
   $$\frac{R_{\varepsilon n}}{\varepsilon n} \to c - \frac{\lambda}{\mu} \qquad \text{almost surely as } n \to \infty,$$
   for any $\varepsilon > 0$ fixed, and hence that $R_{\varepsilon n} \to \infty$ almost surely, if $c > \lambda/\mu$.

   (b) Denote by Y_n, $n \geq 1$, the inter-renewal times of the claims, and define $T_0 = 0$, $T_n = Y_1 + \cdots + Y_n$. Define $S_n = R_{T_n}$ and also consider $R_{T_n-} = S_n + A_n$. Show that
   $$\frac{R_{T_n}}{n} \to \frac{c}{\lambda} - \frac{1}{\mu} \qquad \text{and} \qquad \frac{R_{T_n-}}{n} \to \frac{c}{\lambda} - \frac{1}{\mu} \qquad \text{almost surely as } n \to \infty$$

   (c) Using $R_{T_{N_t}} \leq R_t \leq R_{T_{N_t+1}-}$, deduce from (b) that
   $$\frac{R_t}{t} \to c - \frac{\lambda}{\mu} \qquad \text{almost surely as } t \to \infty.$$

6. Consider a stationary $M/M/1$ queue $(X_t)_{t\geq 0}$ with independent exponential inter-arrival times with rate λ and independent exponential service times with rate μ. Here, the initial distribution is ξ with $\xi_i = \rho^i(1-\rho)$, where $\rho = \lambda/\mu < 1$. Denote by $T_0 = 0$ and $T_{n+1} = \inf\{t > T_n : X_t \neq X_{T_n}\}$ the jump times, by $M_n = X_{T_n}$ the embedded jump chain. Denote by $A_0 = 0$, $A_{m+1} = \inf\{t > A_m : X_t - X_{t-} = 1\}$, $m \geq 0$, the successive arrival times.

   (a) Show that $\mathbb{P}(A_1 > T_k | X_0 = m) = \left(\dfrac{\mu}{\lambda + \mu}\right)^k$, $m \geq k$.

   (b) Show that $\mathbb{P}(A_1 > T_k) = \left(\dfrac{\lambda}{\lambda + \mu}\right)^k$, $k \geq 0$.

   (c) Show that $\mathbb{P}(M_k = i | A_1 > T_k) = \rho^i(1-\rho)$, $i \geq 0$.

   (d) Show that $\mathbb{P}(X_{A_1} = i) = \rho^i(1-\rho)$, $i \geq 2$. Without any further calculations, is ξ stationary for $(X_{A_m})_{m\geq 0}$?

The following questions are meant to deepen your understanding of the earlier material and/or go a little beyond the scope of the course. There will probably not be time for them to be covered in the classes and they will not be marked, but full solutions will be given on the solution sheets.

7. Let $p_n = \frac{\lambda^n}{n!}e^{-\lambda}$, $n \geq 0$, be the probability function of the Poisson distribution. Calculate the associated size-biased distribution. For a random variable X^{sb} with the size-biased distribution, show that $X^{\mathrm{sb}} - 1$ is Poisson distributed.

8. Potential customers arrive at a single-server bank according to a Poisson process $(N_t)_{t\geq 0}$ with rate λ. However, potential customers will enter the bank only if the server is free when they arrive, and otherwise will go home. Assume that the service times are independent random variables with probability density function g and mean ν.

(a) Denote by X_t the number of customers that *have left after completed service* before time t, $t \geq 0$. Show that $(X_t)_{t\geq 0}$ is a renewal process, and describe its inter-renewal distribution.

(b) Calculate the asymptotic rate $\lim_{t\to\infty} X_t/t$ at which customers leave the bank (after completed service).

(c) Consider the proportion $P_t = X_t/N_t$. What long-term proportion of potential customers are actually served?

(d) Consider the sequence of departure times T_n, $n \geq 1$ (departures after completed service). What long-term proportion of time is the server busy?

Hint: Consider this proportion at departure times first and then argue as in the proof of the strong law of renewal theory.

9. In the setting of the previous sheet Question 4 (Proof of the Ergodic Theorem), suppose that the jump chain is also positive recurrent, denote by η its stationary distribution and by $\Pi = (\pi_{i,j})_{i,j\in\mathbb{S}}$ its transition matrix. Let $X_0 = i$.

   Denote by $N(t)$ the number of transitions of X up to time $t \geq 0$, and by $N_i(t)$ the number of transitions to i up to time $t \geq 0$. Show that $\dfrac{N_i(t)}{N(t)} \to \eta_i$ almost surely as $t \to \infty$. Also show that $\dfrac{N_i(t)}{t} \to \dfrac{1}{m_i}$ almost surely as $t \to \infty$. Deduce that $\dfrac{N(t)}{t} \to \dfrac{1}{m_i\eta_i}$ almost surely, and that the limit does not depend on $i \in \mathbb{S}$.

10. Consider a single-server queueing system with Poisson arrivals at rate λ and exponential service times at rate μ. The system has the following special feature: the server can serve two customers at the same time. He can also serve a single customer in the system but then a second customer cannot be jointly served before the single customer leaves. Take $\mathbb{S} = \mathbb{N} \cup \{\emptyset\}$. Let $X_t = \emptyset$ if the server is idle. Let $X_t = 0$ if the server is busy but no-one else is waiting to be served. If the server is busy and there are n people waiting to be served, set $X_t = n$.

    (a) Determine the Q-matrix and the invariant distribution.

    Hint: Try $\xi_n = \alpha^n \xi_0$ for $n \in \mathbb{N}$.

    (b) Determine the long-term proportion of customers that are served alone.

    Hint: Which transitions correspond to a beginning single service? Consider the counting processes counting these transitions separately.

11. Let $\widetilde{X}$ be a delayed renewal process whose first renewal time has density g, the subsequent inter-renewal times density f. Let $F(t) = \int_0^t f(s)ds$, $G(t) = \int_0^t g(s)ds$ and $\overline{F}(t) = 1 - F(t)$, $\overline{G}(t) = 1 - G(t)$.

    (a) Show that $\widetilde{m}(t) = \mathbb{E}(\widetilde{X}_t)$ (as opposed to the undelayed $m(t) = \mathbb{E}(X_t)$) satisfies both

    $$\widetilde{m} = G + m * g \quad \text{and} \quad \widetilde{m} = G + \widetilde{m} * f.$$

    (b) Show, now by conditioning on the last arrival before time t that

    $$\mathbb{P}(\widetilde{E}_t > y) = \overline{G}(t+y) + \int_0^t \overline{F}(t+y-x)\widetilde{m}'(x)dx.$$

    (c) If more specifically $g(y) = f_0(y) = \overline{F}(y)/\mu$, show that $\widetilde{m}(t) = t/\mu$ and that $\widetilde{E}_t$ also has density f_0, for all $t \geq 0$.