B8.3 Mathematical Models for Financial Derivatives

Hilary Term 2020

Problem Sheet One

1. Compute the payoffs and sketch the payoff diagrams for the following portfolios involving European options with expiry date T.

   (a) Long one call and two puts, all with strike $K > 0$ (this is a *strip*).

   (b) Long one put and two calls, all with strike $K > 0$ (this is a *strap*).

   (c) Long one put with strike $K_1 > 0$ and long one call with strike $K_2 > K_1$ (this is a *strangle*).

   (d) Long one call with strike $K_1 > 0$, long one call with strike $K_2 > K_1$ and short two calls with strike $K = (K_1 + K_2)/2$ (this is a *butterfly spread*).

   If $S_0 = K$, where $K = (K_1 + K_2)/2$ in (c) (and (d)), what market outcome at T would a speculator be hoping for in each of these portfolios?

2. In 1986 Standard Oil issued bonds for which the bond holder received no interest but at maturity received the face value of the bond, \$1,000 (this is the amount lent by the bond holder to Standard Oil) plus an additional amount, A, which depended on the price of oil at the bond's maturity. This additional amount was set to be 170 times the excess, if any, of the price of a barrel of oil over \$25 subject to the condition that the additional amount, A, could not exceed \$2550.

   (a) Find a formula for A in terms of S_T, the price of a barrel of oil (in dollars per barrel) at the bond's maturity date T.

   (b) Express the additional amount in terms of a long position in a number of call options with strike K_1 and a short position in a number of call options with strike K_2. (Specify the both the number and strikes of these call options, which are on the price of a barrel of oil.)

3. Let $c(S_t, t; K, T)$ denote the price of a European call option with strike $K > 0$, expiry date T, current time $t < T$ and share price is $S_t > 0$ (assume that $S_t > 0$ implies $S_T > 0$). Use no arbitrage arguments to prove the following properties:

   (a) $c(S_t, t; K, T) \leq S_t$,

   (b) $c(S_t, t; K, T) \geq \max(S_t - K\, e^{-r(T-t)}, 0)$, where r is the risk-free rate,

(c) if $0 < K_1 < K_2$ then

$$0 \le c(S_t, t; K_1, T) - c(S_t, t; K_2, T) \le (K_2 - K_1)\, e^{-r(T-t)},$$

(d) if $T_1 < T_2$ and $r > 0$, then $c(S_t, t; K, T_1) \le c(S_t, t; K, T_2)$.

4. Consider a two-step binomial model in which at each step the share price either doubles, with probability $p \in (0, 1)$, or halves, with probability $1 - p \in (0, 1)$. Initially the price is $S_0 = 4$. Assume each step takes one unit of time and that over one unit of time the risk-free rate is $r = \log(5/4)$. The possible prices are shown in Figure 1 below.

   (a) Show that the risk-neutral probability for an up-move is $q = \frac{1}{2}$.

   (b) Suppose now, and for the rest of the question, that the option has a payoff which depends on the maximum share price over the life of the option, given by

   $$Y = \max_{t=0,1,2} \big(S_t - 6\big)^+.$$

   (This means it is a fixed-strike lookback call.) Compute the final option values $V_2^\omega = Y^\omega$ for each of the outcomes, i.e., possible paths, $\omega \in \Omega = \{uu, ud, du, dd\}$.

   (c) Compute the values of V_1^u and V_1^d, and hence find V_0.

   (d) Find the replicating portfolios $(\phi_1^\omega, \psi_1^\omega)$ for $\omega \in \{u, d\}$.

   (e) Find the replicating portfolio (ϕ_0, ψ_0).

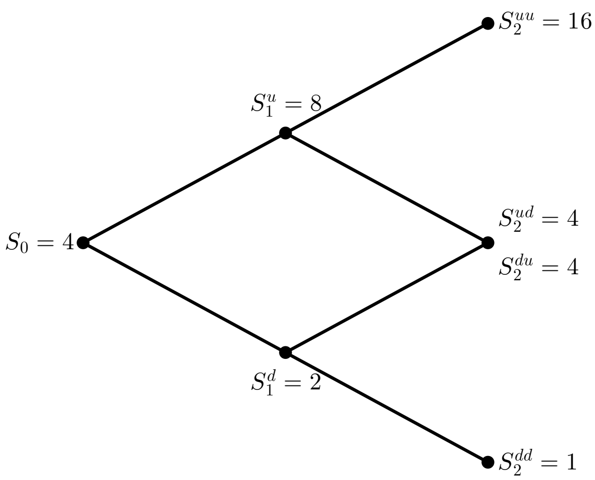

Figure 1: The share-price tree for Questions 4, 5 and 6.

5. Consider a two-step binomial model for a share price with the same properties as in Question 4. An American call is written on this share, with strike $K = 4$. Compute the prices of this American call on the tree and show that they equal the corresponding prices of a European call with strike $K = 4$.

6. With the same two-step binomial model as in the previous two questions, consider the following American option written on the stock. If the option is exercised at time $t \in 0, 1, 2$ it pays out

$$Y_t = \left(\frac{1}{1+t} \Big(\sum_{k=0}^{t} S_k \Big) - 2 \right)^+,$$

i.e., it is an American call with fixed strike $K = 2$ on the *average* share price at time t. Find the value of the option on all paths through the tree and determine the optimal exercise strategy (i.e., find the points on paths in the tree where it is optimal to exercise the option).