B8.3 Mathematical Models for Financial Derivatives

Hilary Term 2020

Problem Sheet Two

In the following $(W_t)_{t\geq 0}$ denotes a standard Brownian motion and $t \geq 0$ denotes time. A partition π of the interval $[0,t]$ is a sequence of points $0 = t_0 < t_1 < t_2 < \cdots < t_n = t$ and $|\pi| = \max_k(t_{k+1} - t_k)$. On a given partition $W_k \equiv W_{t_k}$, $\delta W_k \equiv W_{k+1} - W_k$, $\delta t_k \equiv t_{k+1} - t_k$ and if f is a function on $[0,t]$, $f_k \equiv f(t_k)$ and $\delta f_k \equiv f_{k+1} - f_k$.

1. Show that if $t, s \geq 0$ then $\mathbb{E}[W_s\, W_t] = \min(s,t)$.

2. Suppose we define the following two stochastic integrals, the 'backward-Itô' integral

$$\int_0^t f(W_s, s) \bullet dW_s \;=\; \lim_{|\pi|\to 0} \sum_{k=0}^{n-1} f(W_{k+1}, t_{k+1})(W_{k+1} - W_k),$$

and the Stratonovich integral

$$\int_0^t f(W_s, s) \circ dW_s$$
$$= \lim_{|\pi|\to 0} \sum_{k=0}^{n-1} \tfrac{1}{2}\big(f(W_{k+1}, t_{k+1}) + f(W_k, t_k)\big)\,(W_{k+1} - W_k).$$

Show that

$$2\int_0^t W_s \bullet dW_s = W_t^2 + t, \qquad \mathbb{E}\Big[2\int_0^t W_s \bullet dW_s\Big] = 2\,t,$$
$$2\int_0^t W_s \circ dW_s = W_t^2, \qquad \mathbb{E}\Big[2\int_0^t W_s \circ dW_s\Big] = t$$

3. Assuming that both the integral and its variance exist, show that

$$\mathrm{var}\Big[\int_0^t f(W_s, s)\, dW_s\Big] = \int_0^t \mathbb{E}\big[f(W_s, s)^2\big]\, ds.$$

[Note: if the integral and its variance exist then it is legitimate to interchange the order of expectation and integration.]

4. Use the differential version of Itô's lemma to show that

(a) $\displaystyle\int_0^t W_s\, ds = t\, W_t - \int_0^t s\, dW_s = \int_0^t (t-s)\, dW_s,$

(b) $\displaystyle\int_0^t W_s^2\, dW_s = \tfrac{1}{3} W_t^3 - \int_0^t W_s\, ds$,

(c) $\mathbb{E}\big[e^{aW_t - a^2 t/2}\big] = 1$.

5. Define X_t to be the 'area under a Brownian motion', $X_0 = 0$ and $X_t = \int_0^t W_u\, du$ for $t > 0$. Show that X_t is normally distributed with

$$\mathbb{E}\big[X_t\big] = 0, \quad \mathbb{E}\big[X_t^2\big] = \tfrac{1}{3}\, t^3.$$

Now define Y_t as the 'average area under a Brownian motion',

$$Y_t = \begin{cases} 0 & \text{if } t = 0, \\ X_t / t & \text{if } t > 0. \end{cases}$$

Show that Y_t has $\mathbb{E}[Y_t] = 0$, $\mathbb{E}\big[Y_t^2\big] = t/3$ and that Y_t is continuous for all $t \geq 0$.

Is $\sqrt{3}\, Y_t$ a Brownian motion? Give reasons for you answer.

6. Show that if

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t,$$

and $S_t^* = f(S_t) = S_t^3$, then

$$\frac{dS_t^*}{S_t^*} = 3\,(\mu + \sigma^2)\, dt + 3\sigma\, dW_t.$$

7. Find solutions of the Black–Scholes terminal value problem

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - y)\, S \frac{\partial V}{\partial S} - r\, V = 0, \quad S > 0,\ t < T,$$

$$V(S, T) = f(S), \quad S > 0,$$

when

(a) $f(S) = C$, where C is a constant;

(b) $f(S) = S^\alpha$, where α is a constant.

[Hint: you don't need the Feynman–Kac formula to do either of these. Look for simple functional forms of the solution.]

Optional questions

8. The Black–Scholes equation from a binomial method.

   One step of the Cox, Ross & Rubinstein binomial method can be written as
$$V(S,t) = e^{-r\delta t}\Big(q\,V^u + (1-q)\,V^d\Big)$$
   where
$$V^u = V\big(S^u, t+\delta t\big), \quad V^d = V\big(S^d, t+\delta t\big),$$
$$S^u = S\,e^{\sigma\sqrt{\delta t}}, \quad S^d = S\,e^{-\sigma\sqrt{\delta t}}, \quad q = \frac{e^{r\delta t} - e^{-\sigma\sqrt{\delta t}}}{e^{\sigma\sqrt{\delta t}} - e^{-\sigma\sqrt{\delta t}}},$$
   $\sigma > 0$ is the volatility, r is the risk-free rate and $\delta t > 0$ is the length of the time-step. Supposing this is true for all $S > 0$ and that $V(S,t)$ may be expanded in a Taylor series in both S and t, show that as $\delta t \to 0$
$$\begin{aligned} q &= \frac{1}{2} + \frac{r - \frac{1}{2}\sigma^2}{2\,\sigma}\sqrt{\delta t} + \mathcal{O}(\delta t), \\ V^u &= V + \sqrt{\delta t}\,\sigma\,S\,\frac{\partial V}{\partial S} + \delta t\left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2\Big(S\,\frac{\partial V}{\partial S} + S^2\,\frac{\partial^2 V}{\partial S^2}\Big)\right) + \mathcal{O}(\delta t^{3/2}), \\ V^d &= V - \sqrt{\delta t}\,\sigma\,S\,\frac{\partial V}{\partial S} + \delta t\left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2\Big(S\,\frac{\partial V}{\partial S} + S^2\,\frac{\partial^2 V}{\partial S^2}\Big)\right) + \mathcal{O}(\delta t^{3/2}), \end{aligned}$$
   where V and all its partial derivatives are evaluated at (S,t). Hence show that in the limit $\delta t \to 0$ the option price satisfies the (zero dividend-yield) Black–Scholes equation
$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2\,S^2\,\frac{\partial^2 V}{\partial S^2} + r\,S\,\frac{\partial S}{\partial S} - r\,V = 0.$$

9. The variation of a function, or stochastic process, over $[0,t]$, is
$$\langle f\rangle_t = \lim_{|\pi|\to 0}\sum_{k=0}^{n-1} |\,f_{k+1} - f_k\,|.$$
   If $\langle f\rangle_t$ is finite on $[0,t]$ we say f has bounded variation on $[0,t]$. Show that:

   (a) if f is $C^1[0,t]$ then $\langle f\rangle_t = \int_0^t |f'(t)|\,dt < \infty$;
   (b) if f is a continuous function with $\langle f\rangle_t < \infty$ then its quadratic variation is zero, $[\,f\,]_t = 0$;
   (c) Brownian motion does not have bounded variation;
   (d) the arc length of the graph of a Brownian motion is infinite for any $t > 0$.
   [Hint: if $y = X_t$ has an arc length s then $ds = \sqrt{dy^2 + dx^2} \geq \sqrt{dy^2} = |dy|$.]