B8.3 Mathematical Models for Financial Derivatives

Hilary Term 2020

Problem Sheet Three

Your grade will be determined from the *best five* answers to the first *seven* questions.

1. Assume a zero interest rate, $r = 0$, in this problem (to avoid problems with the time-value of cash payments). Let $0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = t$ be a partition of the interval $[0, t]$. Let $S_u > 0$ be the price of a share at time $u \in [0, t]$, Δ_u be a number of shares at time u and abbreviate $S_{t_k} = S_k$, $\Delta_{t_k} = \Delta_k$. At time $t_0 = 0$ we buy Δ_0 shares, at price S_0, and hold these until time t_1. At time t_1 we buy (or sell) enough shares, at price S_1, so that we have Δ_1 shares, which we hold until time t_2, at which point we buy (or sell) enough shares, at price S_2, so that we have Δ_2 shares. We continue this process until time t_{n-1}, when we end up with Δ_{n-1} shares which we hold until $t_n = t$ at which point we sell all shares we hold, at price S_n. Show that the cost of this procedure is
$$-\sum_{j=0}^{n-1} \Delta_j \, (S_{j+1} - S_j).$$
[Hint: at time step t_{k+1} the change from holding Δ_k shares to holding Δ_{k+1} shares is equivalent to selling all the Δ_k shares and then buying back Δ_{k+1} shares, with both the trades being executed at share price S_{k+1}.]

   Show that, formally at least, in the limit $|\pi| \to 0$ the cost becomes
$$C_t = -\int_0^t \Delta_u \, dS_u$$
   where the integral is an Itô integral (with respect to S_u) and hence deduce that
$$dC_t = -\Delta_t \, dS_t.$$

2. Show that if $V(S, t)$ is a solution of the Black–Scholes equation (for $S > 0$ and $t < T$) then so too are:
   (a) $a\,V(S, t)$ with $a \in \mathbb{R}$;
   (b) $V(bS, t)$ with $b > 0$;
   (c) $a\,V(bS, t)$ with $a \in \mathbb{R}$, $b > 0$.

3. A log-option is an option with the payoff function

$$P_{\mathrm{o}}(S_T) = \log(S_T/K),$$

where the "strike" is positive, $K > 0$. Find the Black–Scholes value function for a European log-option. (Such options are not traded, but they occur in the theory underlying the CBOE's VIX (variance index) which measures the S&P500 index's variance, allowing futures and options to be written on this variance.)

4. Find the Black–Scholes price function of a European digital call option, i.e., an option whose payoff function is

$$f(S_T) = \mathbf{1}_{\{S_T \geq K\}} = \begin{cases} 0 & \text{if } 0 < S_T < K, \\ 1 & \text{if } S_T \geq K. \end{cases}$$

A European digital put option has the payoff $f(S_T) = \mathbf{1}_{\{S_T < K\}}$. Use a no arbitrage argument to establish a digital put-call parity result and hence find the Black–Scholes price function for a digital put.

5. Show that if $V(S,t)$ is a sufficiently differentiable solution of the Black–Scholes equation (for $S > 0$ and $t < T$) then so too is

$$W(S,t) = S\,\frac{\partial V}{\partial S}(S,t).$$

By induction, conclude that if $V(S,t)$ is sufficiently differentiable then

$$\left(S\,\frac{\partial}{\partial S}\right)^n V(S,t), \quad S^n\,\frac{\partial^n V}{\partial S^n}(S,t), \quad n = 2, 3, 4, \ldots$$

are also solutions of the Black–Scholes equation.

6. Let $C_{\mathrm{bs}}(S,t;K,T,r,y,\sigma)$ denote the solution of a Black–Scholes call value problem with strike K, expiry date T, risk-free rate r, continuous dividend yield y and volatility σ. Consider the Black–Scholes problem

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-y)\,S\,\frac{\partial V}{\partial S} - r\,V = 0, \quad S > 0,\ t < T$$

$$V(S,t) = \frac{1}{K^2}\max\bigl(S^3 - K^3, 0\bigr), \quad S > 0.$$

Show that

$$V(S,t) = \frac{1}{K^2} C_{\mathrm{bs}}(S^3, t; K^3, T, r, \hat{y}, \hat{\sigma})$$

where $\hat{y} = 3y - 2r - 3\sigma^2$ and $\hat{\sigma} = 3\sigma$.

[Hint: either write $\hat{S} = S^3$ and do a change of variables in the terminal value problem or think about the payoff and risk-neutral process for $\hat{S}_t = S_t^3$.]

7. Show that if $V(S,t)$ is a solution of the Black–Scholes equation (for $S>0$ and $t<T$) and $B>0$ then

$$W(S,t) = \left(\frac{S}{B}\right)^{2\alpha} V\left(\frac{B^2}{S}, t\right),$$

where $2\,\alpha = 1 - 2(r-y)/\sigma^2)$, is also a solution of the (same) Black–Scholes equation.

[Hint: put $\xi = B^2/S$ and note that $V(\xi,t)$ satisfies the Black–Scholes equation in $\xi > 0$ and $t < T$.]

Optional questions

8. Let $V(S,t)$ satisfy the Black–Scholes problem

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-y)\,S\,\frac{\partial V}{\partial S} - r\,V = 0 \quad S>0,\ t<T,$$
$$V(S,t) = P_{\text{o}}(S), \quad S>0.$$

For some fixed reference price, $S_0 > 0$, set the dimensionless variables $x = \log(S/S_0)$, $\tau = \sigma^2(T-t)$ and $v(x,\tau) = V(S,t)/S_0$. Show that

$$\begin{aligned} \frac{\partial v}{\partial \tau} &= \tfrac{1}{2}\frac{\partial^2 v}{\partial x^2} + k_1 \frac{\partial v}{\partial x} - k_2\, v, \quad x\in\mathbb{R},\ \tau>0, \\ v(x,0) &= p(x), \quad x\in\mathbb{R}, \end{aligned} \tag{1}$$

where k_1 and k_2 are constants which you should find (in terms of r, y and σ) and $p(x)$ is a function which you should also find (in terms of $P_{\text{o}}(S)$).

Assuming that $p(x)$ is a "reasonable" function[1], it can be shown that the solution of (1) is infinitely differentiable in x for $\tau > 0$. Hence deduce that

$$v_n(x,\tau) = \frac{\partial^n v}{\partial x^n}(x,\tau), \quad n = 1,2,3,\ldots$$

are also solution of the partial differential equation in (1) for $\tau > 0$. Infer that if $P_{\text{o}}(S)$ is a "reasonable" function then

$$V_n(S,t) = \left(S\frac{\partial}{\partial S}\right)^n V(S,t), \quad n = 1,2,3,\ldots$$

[1] For example, if $p(x)$ is integrable on every compact subset of $\mathbb{R}$ and there are constants $C>0$ and $\kappa>0$ with $|\,p(x)\,| < C\,e^{\kappa x^2}$ for all x ensures that the solution

$$u(x,\tau) = \frac{1}{\sqrt{2\pi\tau}}\int_{-\infty}^{\infty} p(\xi)\,e^{-(x-\xi)^2/2\tau}\,d\xi$$

is C^∞ in x for $0<\tau<1/2\kappa$.

are also solutions of the Black–Scholes partial differential equation for $t < T$.

9. Show that if we put

$$v(x,\tau) = e^{\alpha\tau+\beta x}\, u(x,\tau),$$

in (1) then, for certain values of α and β, which you should determine, we can reduce (1) to the heat equation problem

$$\begin{aligned} \frac{\partial u}{\partial \tau} &= \tfrac{1}{2}\frac{\partial^2 u}{\partial x^2}, \quad x \in \mathbb{R},\ \tau > 0 \\ u(x,0) &= q(x), \quad x \in \mathbb{R}. \end{aligned} \tag{2}$$

Suppose that $u(x,\tau)$ is the solution to (2) and set $\hat{u}(x,\tau) = u(2b-x,\tau)$ for some constant b. Show that $\hat{u}(x,\tau)$ is also a solution of the heat equation (but not necessarily the initial condition) in (2). Unwinding the transformations that reduced the Black–Scholes equation to the heat equation it is clear that $u(x,\tau)$ leads back to the solution of the original Black–Scholes problem. Show that unwinding the transformations on $\hat{u}(x,\tau)$ leads to the 'reflected' solution

$$\hat{V}(S,t) = S^{2\alpha}\, V\Big(\frac{B^2}{S}, t\Big),$$

where $2\alpha = 1 - 2(r-y)/\sigma^2$ and $B^2 > 0$.

10. The covariation of two functions or processes, X and Y, on $[0,t]$ is defined to be

$$[X,Y]_t = \lim_{|\pi|\to 0} \sum_{k=0}^{n-1} (X_{k+1} - X_k)(Y_{k+1} - Y_k).$$

Show that if both X and Y have finite quadratic variation on $[0,t]$ then $[X,Y]_t$ is finite and satisfies $2\big|\,[X,Y]_t\,\big| \le [X]_t + [Y]_t$.

Assuming $[X]_t$ and $[Y]_t$ are finite, show that

(a) $[X+Y]_t = [X]_t + [Y]_t + 2\,[X,Y]_t$,

(b) $[X,Y]_t = \frac{1}{4}\big([X+Y]_t - [X-Y]_t\big)$.

(c) if X and Y are C^1 functions on $[0,t]$ then $[X,Y]_t = 0$.

11. Let $(W_t)_{t\ge 0}$ and $(Z_t)_{t\ge 0}$ be two Brownian motions. They are correlated with correlation $\rho \in (-1,1)$ if

(a) for all $s,t \ge 0$, $\mathbb{E}\big[(W_{t+s} - W_t)(Z_{t+s} - Z_t)\big] = \rho\, s$,

(b) for all $0 \le p \le q \le s \le t$, the pair $(W_q - W_p)$ and $(Z_t - Z_s)$ are independent and the pair $(W_t - W_s)$ and $(Z_q - Z_p)$ are also independent.

Show that in this case $[W, Z]_t = \rho\, t$, in the sense that

$$\mathbb{E}\big[[W, Z]_t - \rho\, t\big] = 0 \quad \text{and} \quad \mathbb{E}\Big[\big([W, Z]_t - \rho\, t\big)^2\Big] = 0.$$

[Hint: first show that if X and Y are random variables with second moments then $|\,\mathbb{E}[\,XY\,]\,| \le \frac{1}{2}\big(\mathbb{E}[X^2] + \mathbb{E}[Y^2]\big)$.]

[Note that if we define a process by $f_t = f(W_t, Z_t, t)$ where $f(W, Z, t)$ is $C^{2,2,1}$, then (the differential version of) Itô's lemma is

$$\begin{aligned} df_t &= \frac{\partial f}{\partial t}\, dt + \frac{\partial f}{\partial W}\, dW_t + \frac{\partial f}{\partial Z}\, dZ_t \\ &+ \tfrac{1}{2}\frac{\partial^2 f}{\partial W^2}\, d[W]_t + \tfrac{1}{2}\frac{\partial^2 f}{\partial Z^2}\, d[Z]_t + \frac{\partial^2 f}{\partial W \partial Z}\, d[W, Z]_t, \end{aligned}$$

where all functions on the right-hand side are evaluated at (W_t, Z_t, t). The result derived above simplifies this expression.]