B8.3 Week 1 Summary 2020

0.1 Useful textbooks

There are a huge number of books on financial derivatives. Here is a selection, worth consulting for background reading. The numbers in square brackets refer to the bibliography at the end of the notes.

- Steven E. Shreve, *Stochastic calculus for finance I: The binomial asset pricing model*, Springer 2004

  (A superb probabilistic account of the binomial model.)

- Steven E. Shreve, *Stochastic calculus for finance II: Continuous-time models*, Springer 2004

  (A superb first text on stochastic calculus for finance with many examples.)

- Alison Etheridge, *A course in financial calculus*, CUP 2002

  (An excellent primer on stochastic calculus for finance.)

- Paul Wilmott, Sam Howison and Jeff Dewynne, *The mathematics of financial derivatives: A student introduction*, CUP 1995

  (A decent first text on the PDE aspects of the subject.)

- Tomas Björk, *Arbitrage theory in continuous time*, 3rd Ed., OUP 2009

  (A good all-round text which covers many topics outside the scope of the course.)

- Nick H. Bingham and Ruediger Kiesel, *Risk-neutral valuation: Pricing and hedging of financial derivatives*, 2nd Ed., Springer 2004

  (A decent all-round text.)

- Douglas Kennedy, *Stochastic financial models*, CRC Press 2010

  (A good text based on a Cambridge Part III course, with a different emphasis, focusing a little more on portfolio optimisation as opposed to derivative security valuation.)

- Hugo D. Junghenn, *Option valuation: A first course in financial mathematics*, CRC Press 2012 (A good text at about the same level as the course.)

- John C. Hull, *Options, futures and other derivatives*, 8th Ed., Pearson 2011

  (A bestseller that has a more financial as opposed to mathematical bias, and was one of the first textbooks on the subject, becoming a mainstay of many trading rooms.)

- Jean Jacod and Philip Protter, *Probability essentials*, Springer 2003

  (An excellent text on measure-theoretic probability, good for background.)

- Geoffrey R. Grimmett and David R. Stirzaker, *Probability and random processes*, 3rd Ed., OUP 2001

  (An excellent and encyclopedic background probability text.)

1 Aims of the course

Why do we want to study financial modelling? Hasn't an over-reliance on mathematics in finance led to significant social costs?

- Simply ignoring finance isn't going to work.
- Understanding the system, and why it operates the way it does, is the first step to effectively improving it
- Utimately, if we want to act in the real world, economics and finance are going to be part of what we need to do.

What are we trying to do in this course?

- Build financial models and understand how they can be used
- Understand where the models will fail, and where we need to take particular care
- Develop technical proficiency which will allow us to work with better models

2 Assumptions

What are the key concerns of financial modelling?

1. Avoid being exploited.
2. Control, manage and understand your risk.
3. Make a profit (usually by managing risk).

As we will see, the first concern is critical in practice.

We will need some economic assumptions when building financial models. The basic assumptions which underlie the models in the course are:

1. There is a riskless investment (a bank account or bonds) which grows at a constant, continuously compounded rate r. If M_t is invested at time t then it grows to $M_T = M_t\, e^{r(T-t)}$ at time $T > t$. A guaranteed amount of B_T paid at time T is worth $B_t = B_T\, e^{-r(T-t)}$ at time $t < T$. Borrowing and lending rates are both assumed equal to r.

2. There are no trading costs; if an asset can be bought for S_t at time t it can be sold for S_t at time t, and any amount can be bought or sold at the same price.

3. Assets are infinitely divisible, so it possible to own 0.432 shares for instance. This is not a major issue as forwards, calls and puts are usually written on $1,000$s or $10,000$s of shares, rather than one share.

4. Short-selling (i.e. holding negative quantities of an asset) is allowed. This is *often* true.

   In many markets, one can *borrow* assets (for a fixed time, for a fee, which we ignore) and sell them, which allows you to own a negative quantity of the asset. This is known as a *covered* short, and is usually seen as a normal part of a well-functioning market.

   In other markets, you can sell something without owning it, provided you deliver within a specified period (usually 2 days). This is known as a *naked* short, and is quite controversial, as it has been linked to negative effects on asset prices.

The key concept which will alow us to build arguments is arbitrage:

> *An arbitrage is an investment which costs nothing (or less) to set up at time t, $X_t \leq 0$, but at a later time $T > t$ has:*
>
> 1. zero probability of having a negative value, $\mathbb{P}(X_T < 0) = 0$;
> 2. strictly non-zero probability of having a strictly positive value, $\mathbb{P}(X_T > 0) > 0$.

We assume that *no arbitrage opportunities exist.* (In practice they do, but when institutions exploit them supply and demand causes prices to realign in order to eliminate them.)

3 Products

3.1 Forwards

Imagine you have a contract in which you will receive USD in one year, but have to pay costs in GBP at that time. At current exchange rates this deal is profitable, but you are concerned about it becoming unprofitable if there is a fall in USD relative to GBP. How can you manage your risk?

For simplicity, let's assume (for now) that no interest is paid on USD.

A forward is an agreement entered into by two parties at time t in which the holder (who has the long position) promises to pay the agreed forward price $F_t > 0$ for an asset with price S_T at some given maturity date $T > t$, and the writer (who has the short position) promises to deliver the share at time T for the forward price F_t. Both parties are obliged to go through with the transaction regardless of the asset price, $S_T > 0$, at maturity. Under normal circumstances, neither party has to pay to enter the agreement at time t.

Consider first an agreement to sell the asset at time T (so selling our USD); this is known as the *short* position. In this case, the forward may be hedged by borrowing cash equal to S_t, the price of the asset at time t, buying the asset, holding it to maturity then delivering it in return for F_t. The payoff for doing this is $F_t - e^{r(T-t)}S_t$, and has no risk or initial cost. As there must be no arbitrage, we know that

$$F_t - e^{r(T-t)}S_t \leq 0.$$

For the long position, consider short-selling the asset at t, putting the money in a risk-free account and then using the forward to buy back the asset for F_t and close out the short sale. The payoff for doing this is $e^{r(T-t)}S_t - F_t$, and by no arbitrage

$$e^{r(T-t)}S_t - F_t \leq 0$$

As we have assumed no transaction costs, the forward prices on each side of the deal are the same. Therefore, if there is no arbitrage, then

$$F_t = S_t\, e^{r(T-t)}.$$

The payoff diagram for the forward for the long position is a plot of the value of the forward to the holder at maturity against the value of the asset at maturity which is $S_T - F_t$.

Key points:

- The forward value is based on the *current* asset price and the *observed* interest rate. *It does not depend on whether the asset is being fairly priced.*
- We did not need to model the evolution of asset prices in the future.
- If interest rates are positive, and there is no cost/benefit to carrying for the asset (e.g. warehousing costs, foreign interest payments, etc), then the forward price is *above* the current ('spot') price of the asset. (Question: What happens in our argument above if USD pays interest at rate $\hat{r}$?)
- As we approach the expiry date, the forward and spot prices converge.

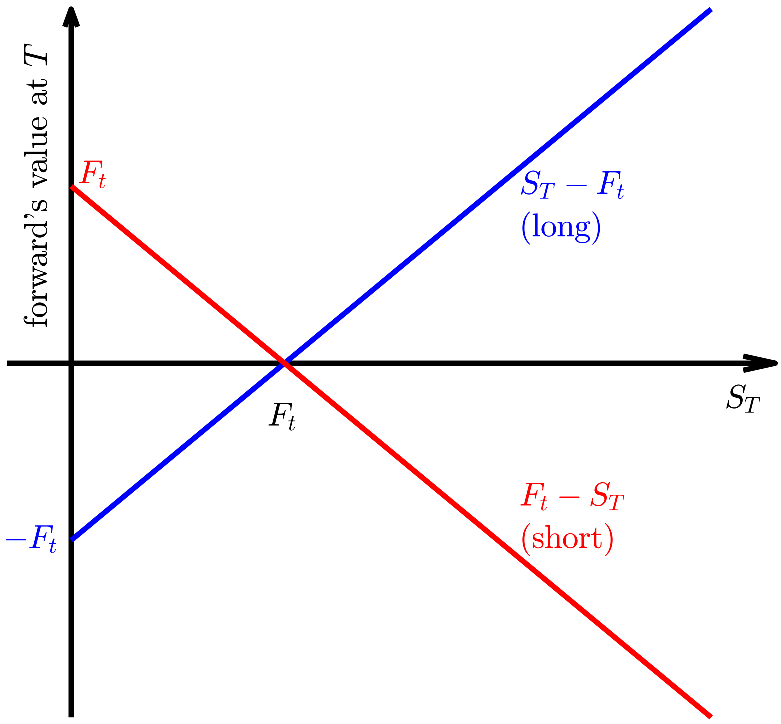

Figure 1: Payoff diagrams for long and short forward positions

- Forwards cost nothing to enter, so provide easy exposure to risk.
- Unlike other many assets we will see, you can't purchase the same forward tomorrow that you purchase today (as the forward price changes).

3.2 Criticism

So, what are the possible flaws in this analysis (in addition to our earlier assumptions)?

- We have ignored any cost/benefit of holding the asset. This is fair enough for stocks or foreign currency (after accounting for dividends and interest), but is difficult for a lot of commoditites, where warehousing is expensive.
- A related issue is that our no-transaction-cost assumption is generally good if the contract is *cash-settled*. If settlement is in real assets, then you may face large transaction costs on the asset side. In practice there may also be a (small) transaction cost in the forward market, so the forward prices available on each side of the deal can be a little different.
- We have assumed that there is no default risk.

The default risk issue is very significant, and has lead to the development of 'Futures' markets. These are very similar to forwards but:

- They have standardized terms (expiry dates, strikes)
- They are traded on an exchange, rather than over the counter
- They are cleared (so your contract is with a clearing house, rather than the person who bought the other side)
- This allows them to be bought and sold freely, as you don't need to keep track of who holds the other side.
- A margin account is needed – this is an account of cash (or other liquid assets), held at the clearing house, which is used to offset changes in the value of your position.

Forward contracts have existed in some form since antiquity – suggestions in the Code of Hammurabi (18th century BC) and in Aristotle's Politics (4th century BC). Formal markets for forwards developed during Tulipmania in Holland in the 1630s. Futures are more recent – the earliest example is the Dojima rice exchange in Osaka, Japan (1697). These became common for agricultural products in the late 19th century (e.g. Chicago Board of Trade 1864 – now part of CME Group), but financial futures (on currencies, interest rates, stock market indices, etc...) were only developed in the 1970s. In many of these markets, futures contracts are the main form of trading in the asset.

3.3 Call options

Suppose we are interested in the fixed-price aspect of a forward, but do not like the risk that we will be out-of-pocket if the asset falls. This leads us into the world of *options*. Options are common on equity (i.e. shares of companies). Let's assume again that a contract is written on a share which pays no dividends and doesn't cost anything to hold.

A call option is a contract with an expiry date $T > t$ and a strike $K > 0$ in which:

1. the holder (who has the long position) has the *right* to buy the underlying share for the strike at the expiry date;
2. the writer (who has the short position) is *obliged* to deliver the share for the strike if the holder exercises their right.

The value (of the long position) of the call at expiry is

$$\max(S_T - K, 0) = (S_T - K)^+$$

and a plot of this function is the call's payoff diagram. Unlike forwards, the holder has to pay a positive amount for the call option (this is the consequence of no arbitrage).

3.4 Put options

A put option is a contract with an expiry date $T > t$ and a strike $K > 0$ in which

1. the holder (who has the long position) has the *right* to sell the underlying share for the strike at the expiry date;
2. the writer (who has the short position) is *obliged* to buy the share for the strike if the holder exercises their right.

The value (of the long position) of the call at expiry is

$$\max(K - S_T, 0) = (K - S_T)^+$$

and a plot of this function is the put's payoff diagram. The holder has to pay a positive amount for a put option (this is the consequence of a no arbitrage argument).

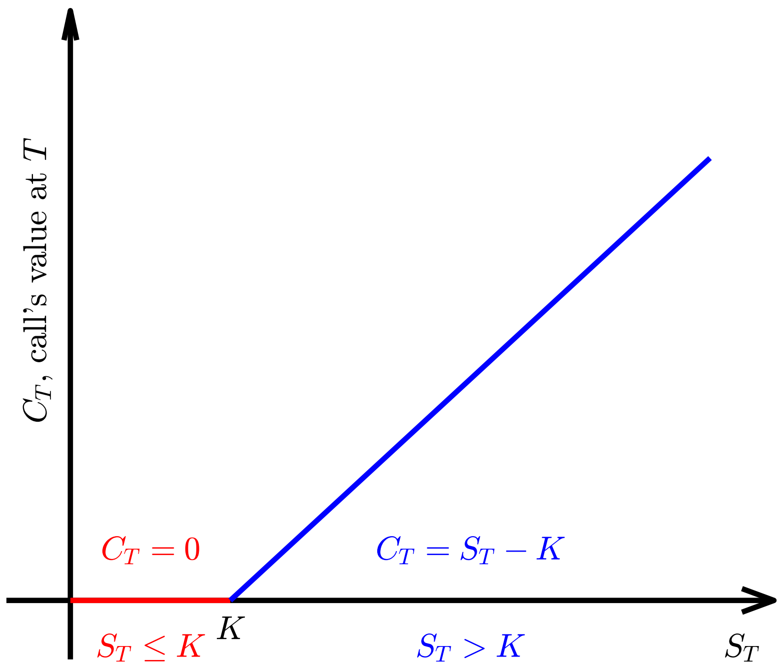

Figure 2: Payoff diagram for a long call position

4 European vs American option

An option which may be exercised *only* at its expiry date is called a European option, one which may be exercised at any time up to and including its expiry date is called an American option. In this course, options will be assumed European unless stated otherwise.

5 Put–call parity

Unlike for forwards, we cannot give a price to an option without building a model for the evolution of stock prices. However, there is a relationship betwen put and call options which must hold.

Let $c(S_t, t; K, T)$ be the price at time t of a European call option with strike K and expiry T and $p(S_t, t; K, T)$ be the price of a European put on the same share and with the same strike and expiry. If a portfolio has one long call and one short put its value at t is

$$c(S_t, t; K, T) - p(S_t, t; K, T)$$

and at expiry its value is

$$(S_T - K)^+ - (K - S_T)^+ = S_T - K.$$

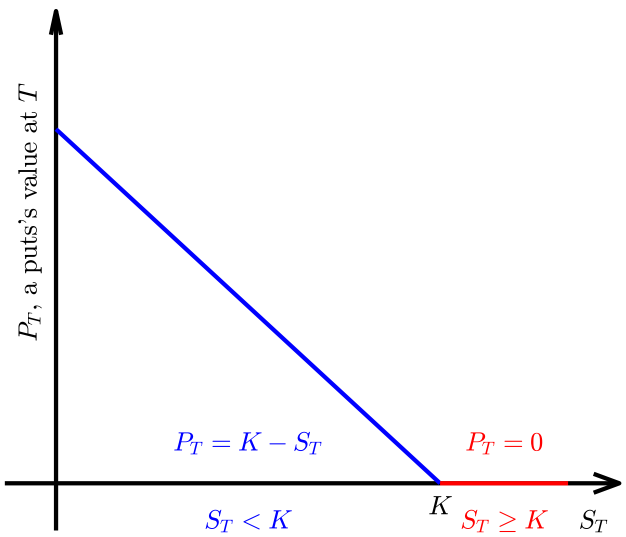

Figure 3: Payoff diagram for a long put position

Consider a portfolio consisting at time t of the asset and a loan of $K\,e^{-r(T-t)}$. At expiry this portfolio also has value $S_T - K$. As the two portfolios also have identical cash flows over $(t, T]$, no arbitrage implies they must have the same value at time t;

$$c(S_t, t; K, T) - p(S_t, t; K, T) = S_t - K\,e^{-r(T-t)}$$

This is known as put-call parity.

6 Law of one price

Put–call parity is a special case of the law of one price; if two portfolios have identical cash flows over the time interval $(t, T]$ and are guaranteed to have the same values at time T, under all possible circumstances, then they must have the same value at time t. If this were not the case, there would be an arbitrage opportunity (at t, short sell the expensive one and buy the cheaper one then close the position out at time T).

7 The simplest model

As we saw, it is not possible to give a price for a call option without a model for future changes in the stock price.

The *simplest* model for a random share price is the one-step binomial model, in which the asset price is S_t at time t. At time T it can be either $S_T = S^u$ with probability $p > 0$ or $S_T = S^d < S^u$ with probability $1-p > 0$. No arbitrage implies that

$$S^d < S_t\, e^{r(T-t)} < S^u.$$

An option with payoff function $f(S_T)$ at time T is written on this asset so at expiry we have

$$\begin{aligned} V_T = V_u &= f(S^u) \quad \text{with probability } p \\ V_T = V_d &= f(S^d) \quad \text{with probability } 1-p \end{aligned}$$

The problem is to find the current value of the option V_t.