B8.3 Week 2 summary 2019

As we saw, it is not possible to give a price for a call option without a model for future changes in the stock price.

The *simplest* model for a random share price is the one-step binomial model, in which the asset price is S_t at time t. At time T it can be either $S_T = S^u$ with probability $p > 0$ or $S_T = S^d < S^u$ with probability $1-p > 0$. No arbitrage implies that

$$S^d < S_t\, e^{r(T-t)} < S^u.$$

An option with payoff function $f(S_T)$ at time T is written on this asset so at expiry we have

$$V_T = V_u = f(S^u) \quad \text{with probability } p$$
$$V_T = V_d = f(S^d) \quad \text{with probability } 1-p$$

The problem is to find the current value of the option V_t. There are at least two ways to do this.

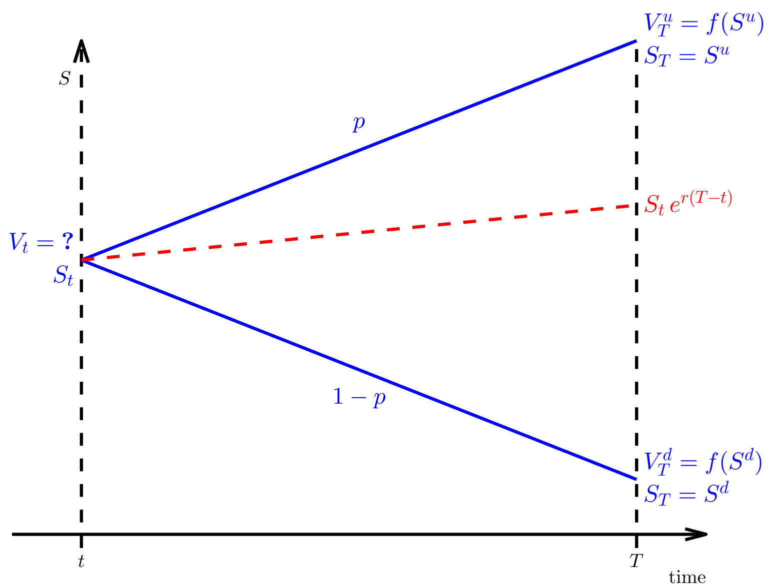

Figure 1: Underlying asset price in a one-step binomial model

Delta hedging argument

At time t set up a portfolio Π long an option and short Δ_t shares

$$\Pi_t = V_t - \Delta_t\, S_t,$$

and hold this portfolio fixed until time T. Choose Δ_t so that the portfolio has the same value regardless of whether the up-state or the down-state occurs, $V^d - \Delta_t\, S^d = V^u - \Delta_t\, S^u$. This gives

$$\Delta_t = \left(\frac{V^u - V^d}{S^u - S^d} \right).$$

This portfolio is *risk-free* and so must grow at the *risk-free rate,* or there would be an arbitrage opportunity. This implies that

$$(V_t - \Delta_t\, S_t)\, e^{r(T-t)} = V^u - \Delta_t\, S^u = V^d - \Delta_t\, S^d$$

and when we solve for V_t we find that

$$V_t = e^{-r(T-t)} \left(q\, V^u + (1-q)\, V^d \right), \quad 0 < q = \left(\frac{S_t\, e^{r(T-t)} - S^d}{S^u - S^d} \right) < 1. \quad (1)$$

Self-financing replication argument

At time t set up a portfolio Φ with ϕ_t shares and ψ_t bonds (bonds grow at the risk-free rate)

$$\Phi_t = \phi_t\, S_t + \psi_t.$$

Hold this portfolio fixed and choose ϕ_t and ψ_t so that the portfolio has value V^u in the up-state and V^d in the down-state

$$\begin{aligned} \Phi^u &= \phi_t\, S^u + \psi_t\, e^{r(T-t)} &= V^u, \\ \Phi^d &= \phi_t\, S^d + \psi_t\, e^{r(T-t)} &= V^d. \end{aligned}$$

Solving for ϕ_t and ψ_t gives

$$\phi_t = \left(\frac{V^u - V^d}{S^u - S^d} \right), \quad \psi_t = \left(\frac{S^u\, V^d - S^d\, V^u}{S^u - S^d} \right) e^{-r(T-t)}.$$

As this portfolio perfectly replicates the option payoff (and has no other cash flows), its value at t must equal V_t. This leads back to (1). (Note that $\Phi \equiv V$, $\psi \equiv \Pi$ and $\phi \equiv \Delta$; either argument amounts to a simple rearrangement of the symbols in the other.)

In this version of the pricing argument we see that *the price of the option is simply the cost of setting up a self-financing portfolio that perfectly covers the option writer's liability at expiry T.*

Interpretation

Note that:

1. no arbitrage on the share price implies that $0 < q < 1$;

2. our market model for the share price is complete in the sense that we can replicate *any* payoff (i.e., solve one equation for Δ_t in the delta-hedging argument or two equations in two unknowns in the replication argument).

3. The number of stocks we hold is 'almost' the derivative of the payoff V with respect to the underlying S (except for discretization).

4. We have not assumed that the stock is being 'fairly' priced, but have found the *only* price which is consistent with the market and does not lead to arbitrage.

As $0 < q < 1$ we can view it as a probability (of an up-jump), the so called *risk-neutral* probability, and write (1) as

$$V_t = \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}\, V_T] = e^{-r(T-t)}\big(qV^u + (1-q)V^d\big). \tag{2}$$

The value of the option at time t is the expected value of option value at expiry, T, under the risk-neutral $\mathbb{Q}$ measure, discounted back to the present via the $e^{-r(T-t)}$ term.

Using the original probabilities p and $1-p$ (the $\mathbb{P}$, or physical, measure) we can define an expected growth rate, μ, for the share by

$$\mathbb{E}^{\mathbb{P}}[S_T] = p\, S^u + (1-p)S^d = S_t\, e^{\mu(T-t)}.$$

Under the $\mathbb{Q}$ measure used to price options in (1) we get

$$\mathbb{E}^{\mathbb{Q}}[S_T] = q\, S^u + (1-q)\, S^d = e^{r(T-t)}\, S_t,$$

so the expected value of the share price grows at the risk-free rate, under the risk-neutral measure, even though the share is *not* risk-free.

Fundamental Theorem of Asset Pricing (simplest case)

There is a fairly general theorem guaranteeing the first equality in (1) holds. The formal statement is this

Fundamental theorem of Asset Pricing

1. Assuming no arbitrage or transaction costs, and deterministic interest rates, there exists a probability measure $\mathbb{Q}$ such that the price of a payoff X_T at time t is given by $e^{-r(t-T)}\mathbb{E}_{\mathbb{Q}}[X_T|\mathcal{F}_t]$. (Also, $\mathbb{Q}$ is equivalent to the real-world probability measure in the sense of measures.)
2. The probability measure $\mathbb{Q}$ is unique if and only if all payoffs are traded (or can be replicated from traded claims).

See Etheridge (2002), §1.5 and §1.6 for a proof in a general discrete time and price model. Here we give a sketch of the proof over a single step, under the assumption that we can trade a claim with any payoff.

Assume no arbitrage or transaction costs. As there are no transaction costs, the prices of all avalable assets must be *linear*, that is, $\Pi(aX+Y) = a\Pi(X) + \Pi(Y)$ for any payoffs X, Y and any constant $a \in \mathbb{R}$. If we assume there are finitely many possible outcomes, then a payoff X can be represented by a vector $(x_1, ..., x_N)$ in $\mathbb{R}^N$ for some N (the number of outcomes). Consequently, we can think of Π as a linear operator mapping $\mathbb{R}^N \to \mathbb{R}$. From algebra, we know that such an operator can always be written as a matrix, in particular,

$$\Pi(X) = \sum_i \pi_i x_i.$$

Considering the case when $x_i \equiv 1$, so the payoff is constant, by no arbitrage we have $\Pi(1) = e^{-rt}$, which implies $\sum_i \pi_i = e^{-rt}$. Write $\pi_i = e^{-rt} q_i$. As we can trade a claim with payoff $(0, .., 0, 1, 0, ...)$, then the price of this claim equals $e^{-rt} q_i$ and no arbitrage guarantees $q_i \geq 0$ and $q_i > 0$ if this outcome happens with nonzero probability.

In general though, this is an *existence* result of a *mathematical fiction* – we do not usually expect $\mathbb{Q}$ to represent real-world probabilities of events.

More than one step

In a multi-step binomial model, we split the interval $[t, T]$ into n steps of length $\delta t = (T - t)/n$, say

$$t_0 = t, \quad t_{m+1} = t_m + \delta t, \quad t_n = T, \quad \text{for} \quad m = 1, 2, \dots n,$$

and build a binomial, or sometimes a binary, tree starting from S_t. It common practice to set

$$S^{\omega u}_{t_{m+1}} = u\, S^{\omega}_{t_m}, \quad S^{\omega d}_{t_m} = d\, S^{\omega}_{t_m},$$

where $u > 1$ and $0 < d < 1$ are constants and, frequently, $u \times d = 1$. Here ω denotes the path to the current node on the tree, for example after two steps $\omega \in \{uu, ud, du, dd\}$. No-arbitrage in the share price tree requires

$$0 < \left(\frac{S^{\omega}_{t_m} e^{r\delta t} - S^{\omega d}_{t_{m+1}}}{S^{\omega u}_{t_{m+1}} - S^{\omega d}_{t_{m+1}}} \right) = \left(\frac{e^{r\delta t} - d}{u - d} \right) < 1.$$

Over each step the risk-neutral pricing formula gives

$$V^{\omega}_{t_m} = e^{-r\delta t} \left(q\, V^{\omega u}_{t_{m+1}} + (1-q)\, V^{\omega d}_{t_{m+1}} \right), \quad q = \left(\frac{e^{r\delta t} - d}{u - d} \right), \tag{3}$$

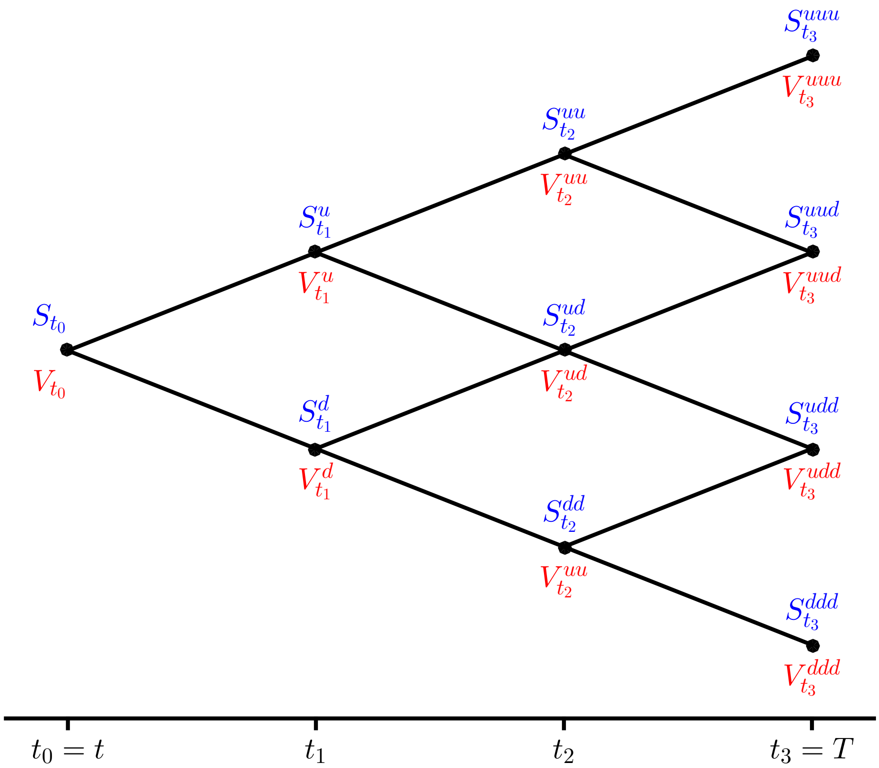

Figure 2: A three-step binomial tree

which requires us to work backwards from $t_n = T$, where we know the option prices from its payoff. This is sometimes called *dynamic programming.*

The Δ-hedging parameter at each step becomes

$$\Delta_{t_m}^{\omega} = \left(\frac{V_{t_{m+1}}^{\omega u} - V_{t_{m+1}}^{\omega d}}{S_{t_{m+1}}^{\omega u} - S_{t_{m+1}}^{\omega d}} \right)$$

and the replicating portfolio (at each step) is

$$\phi_{t_m}^{\omega} = \left(\frac{V_{t_{m+1}}^{\omega u} - V_{t_{m+1}}^{\omega d}}{S_{t_{m+1}}^{\omega u} - S_{t_{m+1}}^{\omega d}} \right), \quad \psi_{t_m}^{\omega} = \left(\frac{S_{t_{m+1}}^{\omega u} V_{t_{m+1}}^{\omega d} - S_{t_{m+1}}^{\omega d} V_{t_{m+1}}^{\omega u}}{S_{t_{m+1}}^{\omega u} - S_{t_{m+1}}^{\omega d}} \right) e^{-r\delta t}.$$

Recall that at time t_m and in state ω, $\phi_{t_m}^{\omega}$ is the number of shares we hold and $\psi_{t_m}^{\omega}$ is the amount of cash hold in order that we perfectly replicate the option's value in the two possible future states.

Self-financing replication

Let S_t be the value of a share and B_t be the value of a bond (i.e., cash) at time t. If at time t a portfolio has ϕ_t shares and ψ_t in cash then the value of the portfolio is

$$\Phi_t = \phi_t \, S_t + \psi_t \, B_t.$$

Let

$$\delta S_t = S_{t+\delta t} - S_t, \quad \delta B_t = B_{t+\delta t} - B_t, \quad \delta \Phi_t = \Phi_{t+\delta t} - \Phi_t$$

so, in general,

$$\begin{aligned} \delta \Phi_t &= \phi_t \, \delta S_t + \psi_t \, \delta B_t \\ &+ (S_t + \delta S_t) \, \delta \phi_t + (B_t + \delta B_t) \, \delta \psi_t \end{aligned}$$

If it turns out that

$$(S_t + \delta S_t) \, \delta \phi_t + (B_t + \delta B_t) \, \delta \psi_t = 0,$$

then any money to buy $\delta \phi_t$ new shares at $t + \delta$ comes from selling $\delta \psi_t$ bonds (i.e., borrowing the same amount of cash) and vice versa. If this is the case, we call the portfolio *self-financing* over $[t, t + \delta t)$ and we find that

$$\delta \Phi_t = \phi_t \delta S_t + \psi_t \delta B_t, \tag{4}$$

which is usually known as the self-financing equation.

The replication strategy given above is self-financing; over any interval $[t_m, t_{m+1})$ both $\phi^{\omega}_{t_m}$ and $\psi^{\omega}_{t_m}$ are fixed, so both $\delta \phi^{\omega}_{t_m} = 0$ and $\delta \psi^{\omega}_{t_m} = 0$. By construction, the replicating portfolio set up at t_m in state ω is guaranteed at time t_{m+1} to have the value of $V^{\omega u}_{t_{m+1}}$ in the up-state (ωu) and $V^{\omega d}_{t_{m+1}}$ in the down-state (ωd). So, although the number of shares and the amount of cash changes from $(\phi^{\omega}_{t_m}, \psi^{\omega}_{t_m})$ to $(\phi^{\omega\, u/d}_{t_m}, \psi^{\omega\, u/d}_{t_m})$ as we go from t^-_{m+1} to t^+_{m+1}, the *value* of the replicating portfolio does not; as we re-adjust the portfolio at t_{m+1}, we sell however many shares are necessary to buy the required number of bonds and vice versa. This establishes that under all possible circumstances in the binomial model, the (ϕ, ψ) strategy both replicates the option's payoff and is self-financing.

American options

At each node on the tree the option holder has two choices:

- hold the option until the next step, in which case its values is given by (3); or
- exercise the option at this step and receive the payoff.

A rational investor will choose the one which makes the option most valuable to them and so if $P^{\omega}_{t_m}$ represents the payoff at the current node then

$$V^{\omega}_{t_m} = \max\Big(e^{-r\delta t} \left(q \, V^{\omega u}_{t_{m+1}} + (1-q) \, V^{\omega d}_{t_{m+1}} \right), \; P^{\omega}_{t_m} \Big) \tag{5}$$

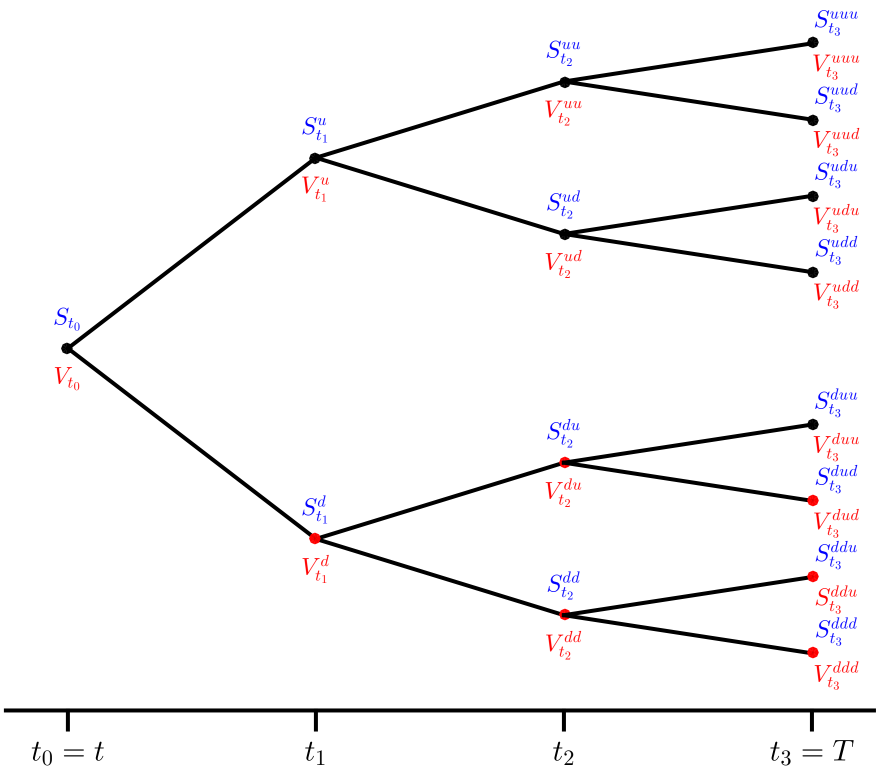

Figure 3: A three-step binary tree: binary trees are sometimes necessary to price *path dependent* options, such as options which depend on the share's average or maximum over the life of the option