B8.3 Week 3 Summary 2020

Brownian motion

A stochastic process is a sequence of random variables indexed by a parameter, for example, $(W_t)_{t\geq 0}$. For each fixed $t \geq 0$, W_t is a random variable.

A process $(W_t)_{t\geq 0}$ is a Brownian motion if (and only if)

1. $\forall\ s \geq 0,\ t \geq 0,\ \big(W_{t+s} - W_t\big)$ is normally distributed with zero mean and variance s,
$$\mathbb{E}[\,W_{t+s} - W_t\,] = 0, \quad \mathbb{E}\big[\,(W_{t+s} - W_t)^2\,\big] = s,$$
2. if $0 \leq p \leq q \leq s \leq t$ then $(W_q - W_p)$ and $(W_t - W_s)$ are independent,
3. the map $t \mapsto W_t$ is continuous, and
4. $W_0 = 0$ (this is really a convention, it saves some writing).

It is not obvious that such a thing exists, but there are a number of ways of constructing it (see Etheridge §3.1 and §3.2, for example, or the most recent B8.2 lectures!).

Note that if $(W_t)_{t\geq 0}$ is a Brownian motion then so too are:

1. $\hat{W}_t = W_{(t+t_0)} - W_{t_0}$ for any constant $t_0 \geq 0$;
2. $\tilde{W}_t = c\,W_{(t/c^2)}$ for any constant $c > 0$.

Brownian motion is almost surely not differentiable

We show that with probability one a Brownian motion is not *differentiable.* If Brownian motion were differentiable at the point $t_0 \geq 0$ then the limit

$$\lim_{t\to 0} \frac{W_{(t+t_0)} - W_{t_0}}{t} = \lim_{t\to 0} \frac{\hat{W}_t}{t}$$

would exist, so it is enough to show that with probability one the second limit does exist. Let A_n and B_n be defined by

$$A_n = \left\{ \frac{|\hat{W}_t|}{t} > n : \text{ for some } t \in \left(0, \frac{1}{n^4}\right] \right\}, \quad B_n = \left\{ \frac{|\hat{W}_t|}{t} > n : \text{ at } t = \frac{1}{n^4} \right\}.$$

Clearly we have $B_n \subseteq A_n$ and so

$$\begin{aligned} \text{prob}(A_n) &\geq \text{prob}(B_n) = \text{prob}\left(\frac{|\hat{W}_{1/n^4}|}{1/n^4} > n \right) \\ &= \text{prob}\left(|n^2\,\hat{W}_{1/n^4}| > \frac{1}{n} \right) \\ &= \text{prob}\left(|\tilde{W}_1| > \frac{1}{n} \right). \end{aligned}$$

As $n \to \infty$ we have $\text{prob}(|\tilde{W}_1| > 1/n) \to 1$. Therefore $\lim_{n\to\infty} \text{prob}(A_n) = 1$ which means that in this limit there is (with probability one) always some $0 < t \leq 1/n^4$ with $|\hat{W}_t|/t > n$. This shows that (with probability one) the limit which defines the derivative of a Brownian motion can not exist.

Quadratic variation

Let π be a partition of $[0, t]$,

$$t_0 = 0 < t_1 < t_2 < \cdots < t_n = t,$$

and let

$$|\pi| = \max_{0 \leq k < n} (t_{k+1} - t_k).$$

We wish to measure the variability of our Brownian motion. This is most easily done by using the 'quadratic variation'[1], which is defined for a random process X_t by:

$$[\,X\,]_t = \mathbb{P}\text{-}\lim_{|\pi|\to 0} \sum_{k=0}^{n-1} (X_{t_{k+1}} - X_{t_k})^2.$$

Importantly, the limit here should be taken *in probability*, in the sense that

$$\mathbb{P}\left[\left|[X]_t - \sum_{k=0}^{n-1} (X_{t_{k+1}} - X_{t_k})^2\right| > \epsilon\right] \to 0 \text{ as } \epsilon \to 0.$$

It may or may not exist, depending on X.

1. If X is continuously differentiable on $[0, t]$ then $[X]_t = 0$.

   As $X_{t_{k+1}} - X_{t_k} = X'(\xi_k)(t_{k+1} - t_k)$ for some $\xi_k \in [t_k, t_{k+1}]$ we have

   $$\begin{aligned} \sum_{k=0}^{n-1} (X_{t_{k+1}} - X_{t_k})^2 &= \sum_{k=0}^{n-1} X'(\xi_k)^2 (t_{k+1} - t_{t_k})^2 \\ &\leq |\pi| \sum_{k=0}^{n-1} X'(\xi_k)^2 (t_{k+1} - t_k) \end{aligned}$$

   and as $|\pi| \to 0$

   $$\sum_{k=0}^{n-1} X'(\xi_k)^2 (t_{k+1} - t_k) \to \int_0^t X'(u)^2 \, du < \infty,$$

   using Riemann's definition of an integral (which is equivalent to Lebesgue's definition if the function is continuous, as it is in this case).

[1] Other common notation for quadratic variation include $[\,X\,]_t$, $\langle X \rangle_t$, $[\,X, X\,]_t$ and $\langle X, X \rangle_t$

2. If X is an increasing continuous function, then $[X]_t = 0$.

   Consider a partition so that $|X_{t_{k+1}} - X_{t_k}| < \epsilon$ for all k. As X is continuous and increasing, such a partition exists and can be taken to have at most $\left\lceil \frac{X_T - X_0}{\epsilon} \right\rceil$ intervals. However, this means that

$$\sum_{k=0}^{n-1} |X_{t_{k+1}} - X_{t_k}|^2 \leq \left\lceil \frac{X_T - X_0}{\epsilon} \right\rceil \epsilon^2 \to 0.$$

3. The quadratic variation of a Brownian motion is defined as

$$[\,W\,]_t = \mathbb{P}\text{-}\lim_{|\pi|\to 0} \sum_{k}^{n-1} (W_{t_{k+1}} - W_{t_k})^2.$$

   Let $\delta W_k = W_{t_{k+1}} - W_{t_k}$ and $\delta t_k = t_{k+1} - t_k$

$$\mathbb{E}\Big[\sum_{k=0}^{n-1}\Big((\delta W_k)^2 - \delta t_k\Big)\Big] = \sum_{k=0}^{n-1}\Big(\mathbb{E}[(\delta W_k)^2] - \delta t_k\Big),$$

   which vanishes for any finite $n > 0$ since $\mathbb{E}[(\delta W_k)^2] = \delta t_k$. It therefore also vanishes in the limit $n \to \infty$.

   Next, consider

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\sum_{k=0}^{n-1}\big((\delta W_k)^2 - \delta t_k\big)\Big)^2\Big] \\
&\quad = \sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\mathbb{E}\big[\big((\delta W_j)^2 - \delta t_j\big)\big((\delta W_k)^2 - \delta t_k\big)\big] \\
&\quad = \sum_{k=0}^{n-1}\mathbb{E}\big[\big((\delta W_k)^2 - \delta t_k\big)^2\big] + \sum_{j\neq k}^{n-1}\mathbb{E}\big[(\delta W_j)^2 - \delta t_j\big]\mathbb{E}\big[(\delta W_k)^2 - \delta t_k\big] \\
&\quad = \sum_{k=0}^{n-1}\mathbb{E}\big[\big((\delta W_k)^2 - \delta t_k\big)^2\big] \\
&\quad = \sum_{k=0}^{n-1}\big(\mathbb{E}[(\delta W_k)^4] - 2\delta t_k\,\mathbb{E}[(\delta W_k^2)] + (\delta t_k)^2\big) \\
&\quad = 2\sum_{k=0}^{n-1}(\delta t_k)^2 \\
&\quad \leq 2\,|\pi|\,\textstyle\sum_{k=0}^{n-1}\delta t_k \;=\; 2\,|\pi|\,t,
\end{aligned}$$

   where we use the independence of δW_j and δW_k if $j \neq k$ to get from the second to the third line.

From this we see that

$$\mathbb{E}\Big[\Big(t-\sum_{k}^{n-1}(W_{t_{k+1}}-W_{t_k})^2\Big)^2\Big]\to 0.$$

so by Markov's inequality, for any $\epsilon>0$,

$$\mathbb{P}\Big(|t-\sum_{k}^{n-1}(W_{t_{k+1}}-W_{t_k})^2|>\epsilon\Big)\le\frac{\mathbb{E}\Big[\Big(t-\sum_{k}^{n-1}(W_{t_{k+1}}-W_{t_k})^2\Big)^2\Big]}{\epsilon^2}\to 0,$$

so the quadratic variation of W is $[W]_t=t$.

It follows that Brownian motion is almost surely *not* continuously differentiable in t.

The Itô integral

The definition of the Itô integral of a function against a Brownian motion is

$$\int_0^t f(W_u,u)\,dW_u=\lim_{|\pi|\to 0}\sum_{k=0}^{n-1}f(W_k,t_k)(W_{t_{k+1}}-W_{t_k}).$$

For fixed t this integral is a random variable and as t varies it is a stochastic process. The sum converges to the integral in an L^2 sense (or in probability, see, e.g., Etheridge pp 78–85).

Using the tower law, and writing $\delta W_k=W_{t_{k+1}}-W_{t_k}$, we find that

$$\begin{aligned}\mathbb{E}\Big[\sum_{k=0}^{n-1}f(W_k,t_k)\delta W_k\Big] &= \mathbb{E}\Big[\sum_{k=0}^{n-1}\mathbb{E}_{t_k}\big[f(W_k,t_k)\delta W_k\big]\Big]\\ &= \mathbb{E}\Big[\sum_{k=0}^{n-1}f(W_k,t_k)\mathbb{E}_{t_k}[\delta W_k]\Big] \;=\; 0.\end{aligned}$$

This establishes that if the Itô integral exists (and f is sufficiently 'nice') then

$$\mathbb{E}\Big[\int_0^t f(W_u,u)\,dW_u\Big]=0.$$

The same sort of argument shows that for $0\le s<t$

$$\mathbb{E}_s\Big[\int_0^t f(W_u,u)\,dW_u\Big]=\int_0^s f(W_u,u)\,dW_u.$$

If f is a reasonable function of t alone then

$$\begin{aligned}
\mathbb{E}\Big[\Big(\sum_{k=0}^{n-1} f(t_k)\,\delta W_k\Big)^2\Big] &= \mathbb{E}\Big[\sum_{j,k=0}^{n-1} f(t_j)\,f(t_k)\,\delta W_k\,\delta W_j\Big] \\
&= \mathbb{E}\Big[\sum_{k=0}^{n-1} \mathbb{E}_k\big[\,f(t_k)^2\,\big(\delta W_k\big)^2\,\big]\Big] + 2\mathbb{E}\Big[\sum_{j<k}^{n-1} \mathbb{E}_k\big[\,f(t_j)f(t_k)\delta W_j\delta W_k\,\big]\Big] \\
&= \mathbb{E}\Big[\sum_{k=0}^{n-1} f(t_k)^2\,\mathbb{E}_k\big[\,\big(\delta W_k\big)^2\,\big]\Big] + 2\mathbb{E}\Big[\sum_{j<k}^{n-1} f(t_j)f(t_k)\delta W_j\mathbb{E}_k\big[\,\delta W_k\,\big]\Big] \\
&= \mathbb{E}\Big[\sum_{k=0}^{n-1} f(t_k)^2\mathbb{E}_{t_k}[(\delta W_k)^2]\Big] \\
&= \sum_{k=0}^{n-1} f(t_k)^2\delta t_k
\end{aligned}$$

and in the limit $|\pi| \to 0$ we obtain *Itô's isometry,*

$$\operatorname{var}\Big[\int_0^t f(u)\,dW_u\Big] = \int_0^t f(u)^2\,du.$$

As the integral is simply the limit of a sum of normally distributed random variables, it is itself normally distributed (proof omitted).

If f depends on W_t the same sort of argument shows that

$$\operatorname{var}\Big[\int_0^t f(W_u,u)\,dW_u\Big] = \int_0^t \mathbb{E}\big[f(W_u,u)^2\big]\,du,$$

provided the right-hand side exists. In general, however, the integral itself is not normally distributed. For example, $2\displaystyle\int_0^t W_u\,dW_u = W_t^2 - t$, which has a χ^2 distribution.

Itô's lemma

If $f(x,\tau)$ is $C^{2,1}$ then

$$f(W_t,t) - f(0,0) =$$

$$\int_0^t \frac{\partial f}{\partial \tau}(W_u,u)\,du + \int_0^t \frac{\partial f}{\partial x}(w_u,u)\,dW_u + \tfrac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(W_u,u)\,d[W]_u. \tag{1}$$

Since $[W]_u = u$ we can replace $d[W]_u$ by du, and in practice we always do.

Consider the simpler case where f is independent of τ and write

$$f(W_t) - f(0) = \sum_{k=0}^{n-1}\big(f(W_{k+1}) - f(W_k)\big)$$

over some partition, π, of $[0,t]$. Taylor's theorem (with remainders) shows that for each k

$$f(W_{k+1}) - f(W_k) = f'(W_k)\delta W_k + \tfrac{1}{2} f''(V_k)(\delta W_k)^2$$

for some V_k between W_k and W_{k+1}, where $\delta W_k = W_{k+1} - W_k$. Thus

$$f(W_t) - f(0) = \sum_{k=0}^{n-1} f'(W_k)\delta W_k + \tfrac{1}{2} \sum_{k=0}^{n-1} f''(V_k)(\delta W_k)^2.$$

As we refine the partition

$$\lim_{|\pi|\to 0} \sum_{k=0}^{n-1} f'(W_k)\delta W_k \to \int_0^t f'(W_u)\, dW_u.$$

For the second sum, it can be shown that

$$\lim_{|\pi|\to 0} \sum_{k=0}^{n-1} f''(V_k)(\delta W_k)^2 \to \int_0^t f''(W_u)\, d[W]_u,$$

establishing that

$$f(W_t) - f(0) = \int_0^t f'(W_u)\, dW_u + \tfrac{1}{2} \int_0^t f''(W_u)\, d[W]_u.$$

Itô's lemma in practice

In practice, we usually write (1) in differential form rather than an integral form. If $f(W,t)$ is $C^{2,1}$ and we define $f_t = f(W_t, t)$ the differential form of Itô's lemma is

$$df_t = \left(\frac{\partial f}{\partial t}(W_t, t) + \tfrac{1}{2} \frac{\partial^2 f}{\partial W^2}(W_t, t) \right) dt + \frac{\partial f}{\partial W}(W_t, t)\, dW_t.$$

This amounts to doing a regular Taylor series expansion of $f(W,t)$ then *pretending* that $dW_t^2 = dt$ (and ignoring terms of higher order than dt).

To solve the stochastic differential equation

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t \tag{2}$$

we can proceed as follows. If $f(W,t) = e^{aW+bt}$ then all its partial derivatives are multiples of the function, so it makes sense to try

$$S_t = S_0\, e^{aW_t + bt}.$$

This gives

$$dS_t = (b\, S_t + \tfrac{1}{2}\, a^2\, S_t)\, dt + a\, S_t\, dW_t$$

or

$$\frac{dS_t}{S_t} = (b + \tfrac{1}{2}\,a^2)\,dt + a\,dW_t.$$

If we set $a = \sigma$ and $b = \mu - \frac{1}{2}\,\sigma^2$ we recover (2), i.e., the solution of (2) is

$$S_t = S_0 \exp\Big((\mu - \tfrac{1}{2}\,\sigma^2)\,t + \sigma\,W_t\Big).$$

The process S_t is often called *geometric* Brownian motion. Note that the sign of S_t is determined by the sign of S_0.

Itô's lemma for solutions of SDEs

Suppose that X_t is a solution of

$$X_t - X_0 = \int_0^t \mu(X_u, u)\,du + \int_0^t \sigma(X_u, u)\,dW_u,$$

$f(x,t)$ is a $C^{2,1}$ function and we set $f_t = f(X_t, t)$. Then

$$f_t - f_0 = \int_0^t \left(\frac{\partial f}{\partial t}(X_u, u) + \tfrac{1}{2}\sigma(X_u, u)^2 \frac{\partial^2 f}{\partial x^2}(X_u, u)\right)\,du + \int_0^t \frac{\partial f}{\partial x}(X_u, u)\,dX_u$$

The proof amounts to showing the quadratic variation $[\,X\,]_t$ is given by

$$[\,X\,]_t = \int_0^t \sigma(X_u, u)^2\,du.$$

In differential notation, which is how this result is normally used, if

$$dX_t = \mu(X_t, t)\,dt + \sigma(X_t, t)\,dW_t \tag{3}$$

and $f_t = f(X_t, t)$ then

$$df_t = \left(\frac{\partial f}{\partial t}(X_t, t) + \tfrac{1}{2}\sigma(X_t, t)^2 \frac{\partial^2 f}{\partial x^2}(X_t, t)\right)\,dt + \frac{\partial f}{\partial x}(X_t, t)\,dX_t. \tag{4}$$

This can be obtained from a Taylor series expansion of $f(x,t)$ and *pretending* that $dX_t^2 = \sigma(X_t, t)^2\,dt$.

The Feynman–Kac formula

Suppose that $f(x,t)$ satisfies the terminal value problem

$$\begin{gathered} \frac{\partial f}{\partial t} + \tfrac{1}{2}\sigma(x,t)^2 \frac{\partial^2 f}{\partial x^2} + \mu(x,t)\,\frac{\partial f}{\partial x} = 0, \quad t < T,\ x \in \mathbb{R}, \\ f(x, T) = F(x), \quad x \in \mathbb{R}. \end{gathered} \tag{5}$$

Let X_t satisfy the stochastic differential equation

$$dX_t = \mu(X_t, t)\, dt + \sigma(X_t, t)\, dW_t$$

Then

$$f(x,t) = \mathbb{E}_t\big[F(X_T) \,|\, X_t = x\big] \tag{6}$$

To see this, note that Itô's lemma implies that

$$\begin{aligned} f(X_T, T) &= f(X_t, t) + \int_t^T \sigma(X_s, s)\, \frac{\partial f}{\partial x}(X_s, s)\, dW_s \\ &+ \int_t^T \Big(\frac{\partial f}{\partial t}(X_s, s) + \mu(X_s, s)\, \frac{\partial f}{\partial x}(X_s, s) + \tfrac{1}{2}\sigma(X_s, s)^2\, \frac{\partial^2 f}{\partial x^2}(X_s, s) \Big)\, ds \end{aligned}$$

By assumption, the integral on the second line vanishes and when we take expectations the integral on the first line also vanishes. Thus

$$f(X_t, t) = \mathbb{E}_t[\, f(X_T, T)\,]$$

and conditioning on $X_t = x$ gives

$$f(x,t) = \mathbb{E}_t[\, f(X_T, T) \,|\, X_t = x\,]$$