B8.3 Week 4 summary 2020

The Black–Scholes model

We will consider a slightly generalized version of the classic Black–Scholes model. Over each infinitesimal period $[t, t+dt)$ the share pays $y\,S_t\,dt$ in dividends, where for our purposes y is a constant known as the continuous dividend yield. This is a poor but widely used model for dividend paying shares.

With reinvestment of dividends, one share at time zero grows to e^{yt} shares at time t and the total value at time t is $p_t = e^{yt}\,S_t$. If we assume that S_t evolves as

$$\frac{dS_t}{S_t} = (\mu - y)\,dt + \sigma\,dW_t, \tag{1}$$

where μ is known as the drift, y is the continuous dividend yield and $\sigma > 0$ is the volatility, then Itô's lemma shows that

$$\frac{dp_t}{p_t} = \mu\,dt + \sigma\,dW_t.$$

If we hold the shares and reinvest the dividends to buy more shares then the value of the holding at time t is p_t and for this reason we write the evolution of S_t as (1), which is equivalent to writing

$$S_t = S_0 \exp\big(\,(\mu - y - \tfrac{1}{2}\sigma^2)\,t + \sigma\,W_t\,\big).$$

For *fixed* $T \geq 0$ the distribution of S_T is given by

$$S_T = S_0 \exp\big(\,(\mu - y - \tfrac{1}{2}\sigma^2)\,T + \sqrt{\sigma^2\,T}\,Z\,\big), \quad Z \sim N(0,1).$$

Delta hedging analysis

Assume an option's payoff is give by $V_T = P_{\mathrm{o}}(S_T)$ and its price $V_t = V(S_t, t)$. Set up a portfolio of one option and $-\Delta_t$ shares, so at t its market price at time t is

$$M_t = V_t - \Delta_t\,S_t.$$

Let Π_t be the cumulative cost of executing this strategy, so

$$d\Pi_t = dV_t - \Delta_t\,dS_t - y\,\Delta_t\,S_t\,dt,$$

the final term represents payment of the dividend yield to the owner of the shares. Itô's lemma applied to $V_t = V(S_t, t)$ gives

$$d\Pi_t = \left(\frac{\partial V}{\partial t}(S_t,t) + \tfrac{1}{2}\sigma^2\,S_t^2\,\frac{\partial^2 V}{\partial S^2}(S_t,t) - y\,\Delta_t\,S_t\right) dt + \left(\frac{\partial V}{\partial S}(S_t,t) - \Delta_t\right) dS_t,$$

which we make (instantaneously) risk-free by setting

$$\Delta_t = \frac{\partial V}{\partial S}(S_t, t).$$

A risk-free portfolio must grow at the risk-free rate, or there would be an arbitrage opportunity, so $d\Pi_t = r\, M_t\, dt$, i.e.,

$$\left(\frac{\partial V}{\partial t}(S_t, t) + \tfrac{1}{2}\sigma^2\, S_t^2\, \frac{\partial^2 V}{\partial S^2}(S_t, t) - y\, S_t\, \frac{\partial V}{\partial S}(S_t, t) \right) = r \left(V_t - S_t \frac{\partial V}{\partial S}(S_t, t) \right),$$

which gives the Black-Scholes equation

$$\begin{array}{ccccc} \underbrace{\frac{\partial V}{\partial t}(S_t, t) + \tfrac{1}{2}\sigma^2\, S_t^2\, \frac{\partial^2 V}{\partial S^2}(S_t, t) - y\, S_t\, \frac{\partial V}{\partial S}(S_t, t)} & + & \underbrace{r\, S_t\, \frac{\partial V}{\partial S}(S_t, t) - r\, V(S_t, t)} & = & 0, \\ \left(\begin{array}{c} \text{rate of return on risk-free} \\ \Delta\text{-hedged portfolio} \end{array} \right) & - & \left(\begin{array}{c} \text{rate of return on} \\ \text{portfolio's value} \\ \text{in bank} \end{array} \right) & = & 0. \end{array}$$

This holds for all *attainable* S_t which, if $S_0 > 0$, is any $S_t > 0$. Thus we obtain the Black-Scholes equation,

$$\frac{\partial V}{\partial t}(S, t) + \tfrac{1}{2}\sigma^2\, S^2\, \frac{\partial^2 V}{\partial S^2}(S, t) + (r - y)\, S\, \frac{\partial V}{\partial S}(S, t) - r\, V(S, t) = 0, \qquad (2)$$

for $S > 0$ and $t < T$. At expiry $V_T = V(S_T, T) = P_o(S_T)$ implies that

$$V(S, T) = P_o(S), \quad S > 0. \qquad (3)$$

Self-financing replication analysis

Here we try to replicate the option's payoff using a portfolio of shares and bonds. The bond price, B_t, evolves as

$$\frac{dB_t}{B_t} = r\, dt. \qquad (4)$$

Let ϕ_t be the number of shares at t and ψ_t be the number of bonds. The market value of the portfolio at t is

$$\Phi_t = \phi_t\, S_t + \psi_t\, B_t \qquad (5)$$

and the change in the portfolio value is

$$d\Phi_t = \phi_t\, dS_t + \psi_t\, dB_t + (S_t + dS_t)\, d\phi_t + (B_t + dB_t)\, d\psi_t + y\, \phi_t\, S_t\, dt,$$

the final term coming from dividends. If $(S_t + dS_t)\, d\phi_t + (B_t + dB_t)\, d\psi_t = 0$ we say the portfolio is self-financing; to buy more shares we have to sell bonds and vice-versa. The self-financing condition is usually written as

$$d\Phi_t = \phi_t\, dS_t + \psi_t\, dB_t + y\, \phi_t\, S_t\, dt.$$

In our case it reduces to

$$d\Phi_t = \phi_t \, dS_t + \psi_t \, r \, B_t \, dt + y \, \phi_t \, S_t \, dt. \tag{6}$$

If we write $\Phi_t = \Phi(S_t, t)$ and apply Itô's lemma we find

$$d\Phi_t = \left(\frac{\partial \Phi}{\partial t}(S_t, t) + \tfrac{1}{2}\sigma^2 \, S_t^2 \, \frac{\partial^2 \Phi}{\partial S^2}(S_t, t) \right) dt + \frac{\partial \Phi}{\partial S}(S_t, t) \, dS_t$$

and matching the deterministic and stochastic terms with those in (6) gives

$$\frac{\partial \Phi}{\partial t}(S_t, t) + \tfrac{1}{2}\sigma^2 \, S_t^2 \, \frac{\partial^2 \Phi}{\partial S^2}(S_t, t) = r \, \psi_t \, B_t + y \, \phi_t \, S_t, \qquad \frac{\partial \Phi}{\partial S}(S_t, t) = \phi_t.$$

Eliminating $\psi_t \, B_t$ using (5) gives

$$\frac{\partial \Phi}{\partial t}(S_t, t) + \tfrac{1}{2}\sigma^2 \, S_t^2 \, \frac{\partial^2 \Phi}{\partial S^2}(S_t, t) = r \, \left(\Phi(S_t, t) - S_t \frac{\partial \Phi}{\partial S}(S_t, t) \right) + y \, S_t \, \frac{\partial \Phi}{\partial S}(S_t, t)$$

for any attainable S_t, i.e., any $S_t > 0$. Rearranging shows that any self-financing portfolio's price function must satisfy

$$\frac{\partial \Phi}{\partial t}(S, t) + \tfrac{1}{2}\sigma^2 \, S^2 \, \frac{\partial^2 \Phi}{\partial S^2}(S, t) + (r - y) \, S \, \frac{\partial \Phi}{\partial S}(S, t) - r \, \Phi(S, t) = 0, \quad S > 0. \tag{7}$$

Finally, we apply the replication condition that the value of the portfolio at T always equals the payoff of the option, i.e.,

$$\Phi(S, T) = P_{\mathrm{o}}(S), \quad S > 0. \tag{8}$$

Then we argue that as the option and the portfolio have exactly the same cash-flows prior to expiry (in both cases here, no cash-flows) and exactly the same values at expiry they must have the same values now, i.e.,

$$V(S, t) = \Phi(S, t).$$

Solution of the Black–Scholes problem

The Black–Scholes problem for the price function of a European option with payoff given by $V_T = P_{\mathrm{o}}(S_T)$ is

$$\begin{gathered} \frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 \, S^2 \, \frac{\partial^2 V}{\partial S^2} + (r - y) \, S \, \frac{\partial V}{\partial S} - r \, V = 0, \quad S > 0, \ t < T, \\ V(S, T) = P_{\mathrm{o}}(S), \quad S > 0. \end{gathered} \tag{9}$$

If we set $V(S, t) = e^{-r(T-t)} \, U(S, t)$ then

$$\begin{gathered} \frac{\partial U}{\partial t} + \tfrac{1}{2}\sigma^2 \, S^2 \, \frac{\partial^2 U}{\partial S^2} + (r - y) \, S \, \frac{\partial U}{\partial S} = 0, \quad S > 0, \ t < T, \\ U(S, T) = P_{\mathrm{o}}(S), \quad S > 0. \end{gathered}$$

and the Feynman Kăc formula shows that

$$U(S,t) = \mathbb{E}_t^{\mathbb{Q}}\big[\, P_{\mathrm{o}}(S_T) \,|\, S_t = S \,\big],$$

where S_t evolves according to

$$\frac{dS_t}{S_t} = (r-y)\,dt + \sigma\, dW_t. \tag{10}$$

Note that this is *not* the same as the SDE for the actual share price, which is (1)—the μ in (1) has become an r in (10).

This means that the option's price can be written as

$$V(S,t) = e^{-r(T-t)}\,\mathbb{E}_t^{\mathbb{Q}}\big[\, P_{\mathrm{o}}(S_T) \,|\, S_t = S \,\big]. \tag{11}$$

We know that if $S_t = S$ then

$$S_T = S\,\exp\big(\,(r-y-\tfrac{1}{2}\sigma^2)\tau + \sigma\, W_\tau\,\big), \quad \tau = T-t$$

and we compute the cumulative distribution function for S_T, for $x>0$, as follows

$$\begin{aligned} F_T(x) &= \mathrm{prob}(S_T < x) \\ &= \mathrm{prob}\big(\log(S_T) < \log(x)\big) \\ &= \mathrm{prob}\Big(\sigma\, W_\tau < \log(x/S) - (r-y-\tfrac{1}{2}\sigma^2)\tau\Big). \end{aligned}$$

As W_τ is $N(0,\tau)$ we can write $\sigma\, W_\tau = \sqrt{\sigma^2\,\tau}\; Z$ where Z is $N(0,1)$, which shows that

$$F_T(x) = \mathrm{prob}\big(Z < d_*\big) = \mathrm{N}(d_*),$$

where

$$d_* = \frac{\log(x/S) - (r-y-\frac{1}{2}\sigma^2)(T-t)}{\sqrt{\sigma^2\,(T-t)}}, \quad \mathrm{N}(d_*) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_*} e^{-p^2/2}\,dp. \tag{12}$$

Differentiating $F_T(x)$ with respect to x gives the probability density function for S_T, conditional on $S_t = S$,

$$f_T(x) = \frac{\exp\big(-\frac{1}{2}d_*^2\,\big)}{x\,\sqrt{2\pi\,\sigma^2\,(T-t)}}, \quad x > 0,$$

and so we arrive at an explicit formula for the option price,

$$V(S,t) = e^{-r(T-t)}\frac{1}{\sqrt{2\pi\,\sigma^2\,(T-t)}}\int_0^\infty P_{\mathrm{o}}(x)\,\exp\big(-\tfrac{1}{2}d_*^2\,\big)\,\frac{dx}{x}, \tag{13}$$

where d_* depends on x (as well as S, $r-y$, σ and $(T-t)$, as in (12)).

General solution of the Black-Scholes problem

The Black-Scholes problem for the price function $V(S,t)$ of a European option with payoff $P_{\mathrm{o}}(S)$ is

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-y)\,S\,\frac{\partial V}{\partial S} - r\,V = 0, \quad S>0,\ t<T,$$

$$V(S,T) = P_{\mathrm{o}}(S), \quad S>0,$$

and the explicit formula for its solution is

$$V(S,t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi\,\sigma^2\,(T-t)}} \int_0^\infty P_{\mathrm{o}}(x)\,\exp\!\big(-\tfrac{1}{2}d_*^2\big)\,\frac{dx}{x}, \tag{14}$$

where d_* is defined as

$$d_* = \frac{\log(x/S) - (r-y-\frac{1}{2}\sigma^2)(T-t)}{\sqrt{\sigma^2(T-t)}.}. \tag{15}$$

The Black-Scholes prices call and put options

The payoff for a call option is $P_{\mathrm{o}}(x) = (x-K)^+ = \max(x-K,0)$ and so the integral in (14) becomes

$$\begin{aligned}
I &= \frac{1}{\sqrt{2\pi\,\sigma^2\,\tau}} \int_0^\infty (x-K)^+ \exp\!\big(-\tfrac{1}{2}d_*^2\big)\,\frac{dx}{x} \\
&= \frac{1}{\sqrt{2\pi\,\sigma^2\,\tau}} \int_K^\infty (x-K) \exp\!\big(-\tfrac{1}{2}d_*^2\big)\,\frac{dx}{x} \\
&= \frac{K}{\sqrt{2\pi\,\sigma^2\,\tau}} \int_0^\infty (e^z-1) \exp\!\big(-\tfrac{1}{2}d_*^2\big)\,dz \qquad (x = K\,e^z).
\end{aligned}$$

In terms of z and $\tau = T-t$ we find that

$$d_* = \frac{z}{\sqrt{\sigma^2\tau}} - d_-, \quad d_- = \frac{\log(S/K) + (r-y-\frac{1}{2}\sigma^2)\tau}{\sqrt{\sigma^2\,\tau}}$$

which suggests the substitution $\zeta = z/\sqrt{\sigma^2\,\tau}$. This gives

$$\begin{aligned}
I &= \frac{K}{\sqrt{2\pi}} \left(\int_0^\infty e^{\sqrt{\sigma^2\tau}\,\zeta}\, e^{-\frac{1}{2}(\zeta-d_-)^2}\,d\zeta - \int_0^\infty e^{-\frac{1}{2}(\zeta-d_-)^2}\,d\zeta \right) \\
&= \frac{K}{\sqrt{2\pi}} \left(e^{\frac{1}{2}(d_+^2-d_-^2)} \int_0^\infty e^{-\frac{1}{2}(\zeta-d_+)^2}\,d\zeta - \int_0^\infty e^{-\frac{1}{2}(\zeta-d_-)^2}\,d\zeta \right),
\end{aligned}$$

where

$$d_+ = d_- + \sqrt{\sigma^2\tau} = \frac{\log(S/K) + (r-y+\frac{1}{2}\sigma^2)\tau}{\sqrt{\sigma^2\,\tau}},$$

$$\tfrac{1}{2}\big(d_+^2 - d_-^2\big) = \log(S/K) + (r-y)\tau.$$

Therefore

$$\begin{aligned} I &= \frac{S\,e^{(r-y)\tau}}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{1}{2}(\zeta-d_+)^2}\,d\zeta - \frac{K}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{1}{2}(\zeta-d_-)^2}\,d\zeta \\ &= \frac{S\,e^{(r-y)\tau}}{\sqrt{2\pi}} \int_{-\infty}^{d_+} e^{-\frac{1}{2}\xi^2}\,d\xi - \frac{K}{\sqrt{2\pi}} \int_{-\infty}^{d_-} e^{-\frac{1}{2}\xi^2}\,d\xi \qquad (\xi = d_\pm - \zeta) \\ &= e^{r\tau}\Big(S\,e^{-y\tau}\,\mathrm{N}(d_+) - K\,e^{-r\tau}\,\mathrm{N}(d_-)\Big). \end{aligned}$$

Multiplying by $e^{-r\tau}$ gives the celebrated Black-Scholes formula for the price (function) of a European call,

$$C(S,t) = S\,e^{-y(T-t)}\,\mathrm{N}(d_+) - K\,e^{-r(T-t)}\,\mathrm{N}(d_-), \tag{16}$$

where

$$d_\pm = \frac{\log(S/K) + (r - y \pm \frac{1}{2}\sigma^2)(T-t)}{\sqrt{\sigma^2(T-t)}}, \quad \mathrm{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}p^2}\,dp.$$

The Δ for a call option is $\Delta_c(S,t) = (\partial C/\partial S) = e^{-y(T-t)}\,\mathrm{N}(d_+)$.