# B8.3 Week 5 summary 2020

## The Black-Scholes prices of European call options

The explicit formula for the price of a European option is

$$V(S,t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi\,\sigma^2\,(T-t)}} \int_0^\infty P_o(x)\,\exp\bigl(-\tfrac{1}{2}d_*^2\bigr)\,\frac{dx}{x}, \tag{1}$$

where $d_*$ is defined as

$$d_* = \frac{\log(x/S) - (r - y - \frac{1}{2}\sigma^2)(T-t)}{\sqrt{\sigma^2(T-t)}.}. \tag{2}$$

The payoff for a call option is $P_o(x) = (x-K)^+ = \max(x-K, 0)$ and so the integral in (1) becomes

$$\begin{aligned} I &= \frac{1}{\sqrt{2\pi\,\sigma^2\,\tau}} \int_0^\infty (x-K)^+\,\exp\bigl(-\tfrac{1}{2}d_*^2\bigr)\,\frac{dx}{x} \\ &= \frac{1}{\sqrt{2\pi\,\sigma^2\,\tau}} \int_K^\infty (x-K)\,\exp\bigl(-\tfrac{1}{2}d_*^2\bigr)\,\frac{dx}{x} \\ &= \frac{K}{\sqrt{2\pi\,\sigma^2\,\tau}} \int_0^\infty (e^z - 1)\,\exp\bigl(-\tfrac{1}{2}d_*^2\bigr)\,dz \qquad (x = K\,e^z). \end{aligned}$$

In terms of $z$ and $\tau = T - t$ we find that

$$d_* = \frac{z}{\sqrt{\sigma^2\tau}} - d_-, \quad d_- = \frac{\log(S/K) + (r - y - \frac{1}{2}\sigma^2)\tau}{\sqrt{\sigma^2\,\tau}}$$

which suggests the substitution $\zeta = z/\sqrt{\sigma^2\,\tau}$. This gives

$$\begin{aligned} I &= \frac{K}{\sqrt{2\pi}} \left( \int_0^\infty e^{\sqrt{\sigma^2\tau}\,\zeta}\,e^{-\frac{1}{2}(\zeta - d_-)^2}\,d\zeta - \int_0^\infty e^{-\frac{1}{2}(\zeta-d_-)^2}\,d\zeta \right) \\ &= \frac{K}{\sqrt{2\pi}} \left( e^{\frac{1}{2}(d_+^2 - d_-^2)} \int_0^\infty e^{-\frac{1}{2}(\zeta-d_+)^2}\,d\zeta - \int_0^\infty e^{-\frac{1}{2}(\zeta-d_-)^2}\,d\zeta \right), \end{aligned}$$

where

$$d_+ = d_- + \sqrt{\sigma^2\tau} = \frac{\log(S/K) + (r - y + \frac{1}{2}\sigma^2)\tau}{\sqrt{\sigma^2\,\tau}},$$

$$\tfrac{1}{2}\bigl(d_+^2 - d_-^2\bigr) = \log(S/K) + (r-y)\tau.$$

Therefore

$$\begin{aligned} I &= \frac{S\,e^{(r-y)\tau}}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{1}{2}(\zeta-d_+)^2}\,d\zeta - \frac{K}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{1}{2}(\zeta-d_-)^2}\,d\zeta \\ &= \frac{S\,e^{(r-y)\tau}}{\sqrt{2\pi}} \int_{-\infty}^{d_+} e^{-\frac{1}{2}\xi^2}\,d\xi - \frac{K}{\sqrt{2\pi}} \int_{-\infty}^{d_-} e^{-\frac{1}{2}\xi^2}\,d\xi \qquad (\xi = d_\pm - \zeta) \\ &= e^{r\tau}\Bigl(S\,e^{-y\tau}\,\mathrm{N}(d_+) - K\,e^{-r\tau}\,\mathrm{N}(d_-)\Bigr). \end{aligned}$$

Multiplying by $e^{-r\tau}$ gives the celebrated Black-Scholes formula for the price (function) of a European call,

$$C(S,t) = S\, e^{-y(T-t)}\, \mathrm{N}(d_+) - K\, e^{-r(T-t)}\, \mathrm{N}(d_-), \tag{3}$$

where

$$d_\pm = \frac{\log(S/K) + (r - y \pm \frac{1}{2}\sigma^2)(T-t)}{\sqrt{\sigma^2(T-t)}}, \quad \mathrm{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}p^2}\, dp.$$

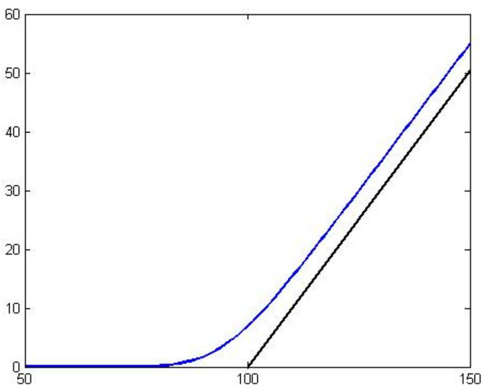


Figure 1: Call price vs Payoff in BS model

The $\Delta$ for a call option is $\Delta_c(S,t) = (\partial C/\partial S) = e^{-y(T-t)}\, \mathrm{N}(d_+)$.
To see this, first note that

$$\frac{\partial d_+}{\partial S} = \frac{\partial d_-}{\partial S} = \frac{1}{S\sqrt{\sigma^2(T-t)}}$$

and from the normal density,

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2},$$

in particular, using the above calculation for $d_+^2 - d_-^2$,

$$N'(d_+) = \frac{1}{\sqrt{2\pi}} e^{-d_+^2/2} = \frac{1}{\sqrt{2\pi}} e^{-d_-^2/2} e^{\log(S/K)+(r-y)\tau} = \frac{Se^{-y\tau}}{Ke^{-r\tau}} N'(d_-).$$

Then apply the chain and product rules, and simplify,

$$\begin{aligned}
\frac{\partial C}{\partial S} &= e^{-y(T-t)}\, \mathrm{N}(d_+) + S\, e^{-y(T-t)}\, \mathrm{N}'(d_+) \frac{\partial d_+}{\partial S} - K\, e^{-r(T-t)}\, \mathrm{N}'(d_-) \frac{\partial d_-}{\partial S} \\
&= e^{-y(T-t)}\, \mathrm{N}(d_+) + \Big(S\, e^{-y(T-t)}\, \mathrm{N}'(d_+) - K\, e^{-r(T-t)}\, \mathrm{N}'(d_-)\Big) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \\
&= e^{-y(T-t)}\, \mathrm{N}(d_+)
\end{aligned}$$

## European Put options

The price of a European put option follows from put-call parity,

$$C(S,t) - P(S,t) = S\,e^{-y(T-t)} - K\,e^{-r(T-t)},$$

and is

$$P(S,t) = K\,e^{-r(T-t)}\,\mathrm{N}(-d_-) - S\,e^{-y(T-t)}\,\mathrm{N}(-d_+). \tag{4}$$

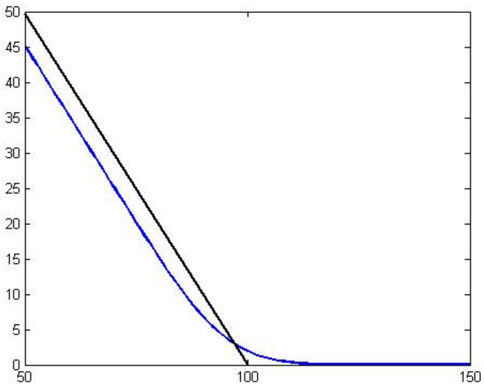


Figure 2: Put price vs Payoff in BS model

The put's delta (most easily found by differentiating put-call parity with respect to $S$) is $\Delta_p(S,t) = (\partial P/\partial S) = -e^{-y(T-t)}\mathrm{N}(-d_+)$.

Some other properties can be observed also:

- For a call option, as $S \to \infty$, we have

$$d_+ \to \infty \qquad \text{and} \qquad d_- \to \infty,$$

  so $C^{BS} \approx S_t e^{y(T-t)} - K e^{r(T-t)}$. This is natural, as the call we can be sure that the call will be exercised, so its value is similar to the value of a forward with the corresponding strike. Conversely, the put option's price converges to zero, for the same reason.

- As $S \to 0$ we have $d_\pm \to -\infty$, so the call price converges to zero, and the put price converges to $K e^{r(T-t)} - S_t e^{y(T-t)}$, which is the price of the short forward with strike $K$.

- As $t \to T$, we have

$$d_\pm \to \begin{cases} \infty & \text{if } S > K \\ -\infty & \text{if } S < K \end{cases}$$

  so $C^{BS} \to (S-K)^+$, that is, the price of the option converges to its payoff.

## Implied volatility.

How do we choose the value of $\sigma$?

- Estimation from historical data:

$$\frac{1}{(N-1)\delta t}\sum_{i=1}^{N}\left(\frac{S_{i+1}-S_i}{S_i}\right)^2.$$

  This is natural. However this method has certain drawbacks due to the fact that, in reality, $\sigma$ varies significantly over time, and it is rather hard to capture its "most recent" value: the estimator becomes less reliable as we decrease $N$, while, if we increase it, we obtain an "averaged" value of $\sigma$ over time.

- Another way is to deduce $\sigma$ from the prices currently observed in the market. This gives rise to **Implied Volatility**: given the market price of an option, find the volatility for which the BS price coincides with the market price:

$$V_t^{mrkt} = V^{BS}(S_t, t; \sigma_{imp})$$

Implied Volatility is defined as a solution to the above equation, where the options in question are the European call (or put) options.

Implied Volatility is well defined, provided

- $V_t^{call,mrkt}$ is **not impossible**: $V_t^{call,mrkt} \in \left[\left(S - Ke^{-r(T-t)}\right)^+, S\right]$.
- And $\frac{\partial}{\partial\sigma}V^{call,BS}$ **does not change its sign**, as a function of $\sigma$.

The first condition is satisfied, since otherwise there is a **model-independent arbitrage**.

The second condition is satisfied, since, as we'll see, the BS Vega is always nonnegative.

## Implied Smile.

If the BS model was true, there would exist one value of implied volatility $\sigma_{imp}$ for call options of all strikes and maturities.

However, **this is not true** in practice. Typically, for each pair $(T, K)$, we have a different value of implied vol $\sigma_{imp}(T, K)$.

Plotted as a function of *negative log-moneyness* $x = \log(K/S)$, this function is typically *convex* around $x = 0$, and, hence, is often referred to as the **implied smile**.

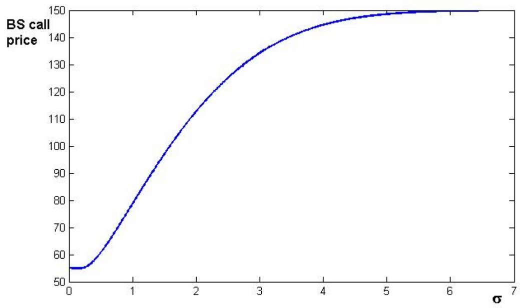


Figure 3: Black-Scholes price of a call option with strike $K = 100$ and underlying level $S = 150$, as a function of the volatility $\sigma$

**Remark 1** *Log-moneyness of a call or put option is defined as* $\log(S/K)$.

In equities (where $S$ is the price of a stock or stock index), the implied smile typically has a **negative skew**, assigning higher values to negative $x = \log(K/S)$ (i.e. $K < S$).

The general (heuristic) explanations for the presence of skew are all related to the fact that people tend to **overestimate** the **risks of extreme negative events**.

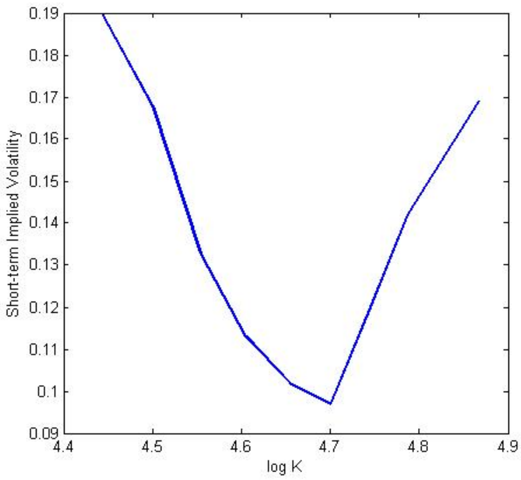


Figure 4: Implied volatility of options on the SP500 index, for the smallest available maturity, plotted as a function of log-strike.

## Greeks and Hedging.

Assume the market is described by a BS model, and denote the price function of an option by $V(S,t)$.

**Sensitivities** of the option price $V$ with respect to the input **variables** ($S$ and $t$) and **parameters** ($\sigma$ and $r$) are called the **Greeks**.

These sensitivities are very important for **hedging** and **risk management**, as they show how the value of the option changes with small changes in the **uncertain** input!

## Delta

**Delta** is defined as

$$\Delta = \frac{\partial}{\partial S} V,$$

and it is the primary sensitivity, as, even if the model is true, the value of underlying will change, and its change is likely to be of a **higher magnitude than the time increment**.

**Remark 2** *Notice that*

$$S_{t+\delta t} \approx S_t \mu \delta t + S_t \sigma \xi \sqrt{\delta t},$$

*where $\xi$ is a standard normal.*

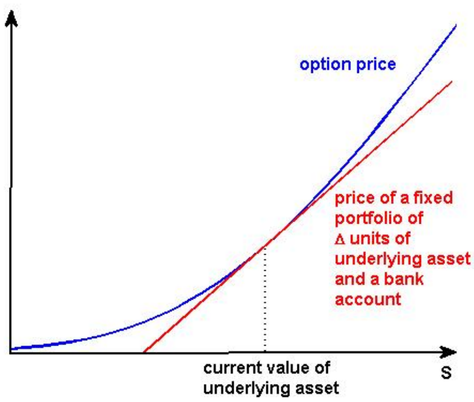


Figure 5: Option price as a function of the underlying level $S$, and its tangent line which represents the value of a portfolio consisting of $\Delta$ units of underlying and the right amount of money in bonds

For a call option $\Delta^{call} = N(d_+)$

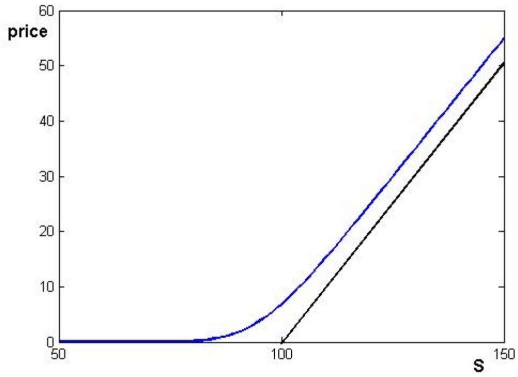


Figure 6: Black-Scholes price of a call option with strike $K = 100$ as a function of the underlying level $S$

By put-call parity, $\Delta^{put} = \Delta^{call} - 1 = -N(-d_+)$

The portfolio consisting of $\Delta_t$ units of $S_t$ and $V_t - \Delta_t S_t$ units of wealth in the bank account is an "instantaneous perfect hedge" of a short position in the option, in the BS model.

It is still a **reasonably good hedge in other models**.

**Example 1** *Consider a call option with* $r = 0.05$, $\sigma = 0.2$, $T = 1$, $K = 100$. *If* $S = 100$, *then* $V = 10.396$ *and* $\Delta = 0.635$. *We open the Delta-hedge, and the hedged portfolio consists of*

- $\Delta_t S_t$ *invested in stock,*
- $-V_t$ *in the option*
- *and* $V_t - \Delta_t S_t$ *in bonds*

*In total, we have* $10.396 - 0.635 \cdot 100 = -53.11$ *invested in stock and option, and* $53.11$ *in bonds.*

*Suppose, on the next day* $S = 101$. *The value of the total position in stock and option becomes* $11.013 - 0.635 \cdot 101 = -53.12$. *The increment from the bond position is negligible. It is reasonable to measure the error of the hedge (which is due to "freezing" the hedging weights) by the total value of all positions at the final time. Then, the error of the hedge relative to the increment of the underlying asset is of the order* $10^{-2}$.

In practice, we can only trade finitely often.

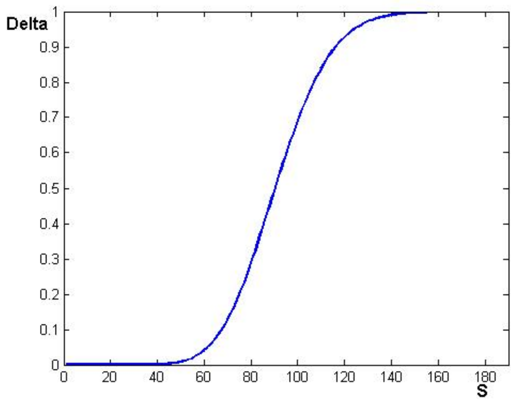


Figure 7: Black-Scholes Delta of a call option with strike $K = 100$ as a function of the underlying level $S$

As a result, we encounter the **discretization error** – the price of the hedging (replicating) portfolio no longer coincides with the option price at all times.

In particular, we **cannot keep both positions** – in the stock and in bonds – **as prescribed by the BS model**.

Therefore, at each moment of rebalancing, we have to choose whether

- we keep $\Delta$ (the amount of shares of stock) as prescribed by the model, and invest the rest of the available capital in bonds (or borrow the required amount by shorting),
- or keep $\gamma_t B_t$ (the amount of money in bonds) as prescribed by the model, and invest the rest in the stock.

Typically traders choose to keep the value of $\Delta$ as prescribed by the model, because changes in the stock price are more significant than changes in the value of bonds.

Such strategy is called **Delta-hedging**.

The nature of the discretization error is explained in Figure 8.

## Gamma

**Gamma** is the sensitivity of $\Delta$ with respect to changes in $S$:

$$\Gamma = \frac{\partial}{\partial S}\Delta = \frac{\partial^2}{\partial S^2}V.$$

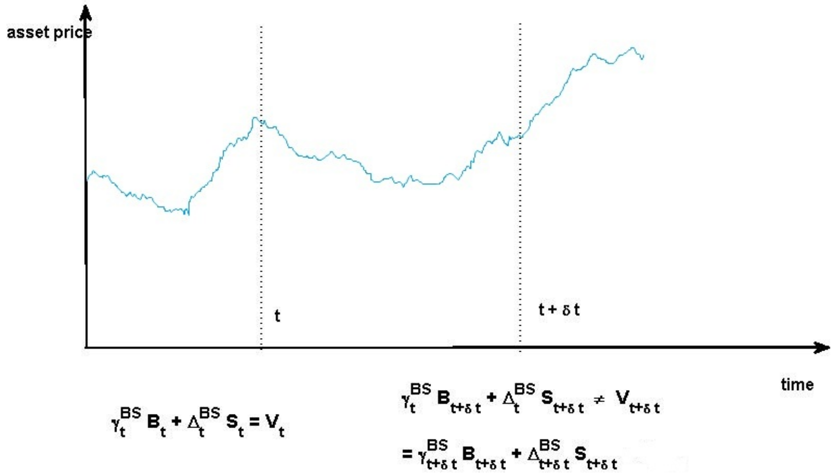


Figure 8: Discretization error of the Delta-hedging. The hedging portfolio is set up at time $t$, however, at time $t+\delta t$, since it has not been updated dynamically in the time interval $[t, t+\delta t]$, the value of the hedging portfolio no longer coincides with the option price. Thus, in order to stay self-financing, we cannot choose both $\gamma_{t+\delta t}$ and $\Delta_{t+\delta t}$ as prescribed by the BS model.

It measure how fast the hedging weight $\Delta$ changes with the changes in the underlying.

This is important since, as mentioned above, in practice we only trade at discrete times.

It is clear from Figure 5 that the smaller is the curvature of the price of an option, as a function of $S$, the smaller is the error of the discretization hedge – the difference between the price function and its tangent line around the point of tangency.

Thus, **Gamma measures the discretization error**.

Let's see make this statement more precise. Assume we have short-sold an option and set up the hedging portfolio at time $t$:

$$\gamma_t = V_t - \Delta_t S_t, \qquad \Delta_t = \frac{\partial}{\partial S} V(S_t, t)$$

Then the hedging error at time $t + \delta t$ is given by

$$
\begin{aligned}
&\gamma_t B_{t+\delta t} + \Delta_t S_{t+\delta t} - V(S_{t+\delta t}, t+\delta t) \\
&= (V(S_t,t) - \Delta_t S_t)(1 + r\delta t) + \Delta_t S_{t+\delta t} - V(S_{t+\delta t}, t+\delta t) \\
&= V_t - S_t \frac{\partial}{\partial S} V_t + r\delta t (V_t - S_t \frac{\partial}{\partial S} V_t) \\
&\qquad - V_t - \delta t \frac{\partial}{\partial t} V_t - (S_{t+\delta t} - S_t) \frac{\partial}{\partial S} V_t \\
&\qquad - \frac{1}{2}\sigma^2 S_t^2 (W_{t+\delta t} - W_t)^2 \frac{\partial^2}{\partial S^2} V_t + S_{t+\delta t} \frac{\partial}{\partial S} V_t + o(\delta t) \\
&= -\frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2}{\partial S^2} V_t ((W_{t+\delta t} - W_t)^2 - \delta t) + o(\delta t) \\
&= -\frac{1}{2}\sigma^2 S_t^2 \Gamma_t ((W_{t+\delta t} - W_t)^2 - \delta t) + o(\delta t),
\end{aligned}
$$

where $o(\delta t)$ is a function satisfying

$$
\frac{o(\delta t)}{\delta t} \to 0,
$$

as $\delta t \to 0$.

Notice that $(\delta W_t)^2 - \delta t$ is a random variable with zero mean and variance $2(\delta t)^2$. It is easy to see that the average absolute value of this term is of the order const $\delta t$.

We conclude $\Gamma_t$ scales the main term in the discretization error of the hedge.

If $\Gamma_t < 0$ (short Gamma), the hedged portfolio benefits from large market moves, and looses on small ones.
If $\Gamma_t > 0$ (long Gamma) – vice versa.

If we hedge a long position in the option, the opposite conclusions hold.

**Remark 3** *Heuristics: even though the volatility in BS model is constant, one can think of an approximate volatility that results from estimating the price returns, or, equivalently, the magnitude of $W_{t+\delta t} - W_t$. In that case, if we (somehow) knew that the period of high magnitude of $W_{t+\delta t} - W_t$, predicted by high estimated volatility, is coming, we could make money by hedging an option with short gamma.*

Gamma of a call is

$$
\Gamma^{call} = \frac{e^{-\frac{1}{2}d_+^2}}{S\sigma\sqrt{2\pi(T-t)}}
$$

As $t \to T$, $\Gamma^{call} \to \frac{\partial^2}{\partial S^2}(S-K)^+ = \delta(S-K)$

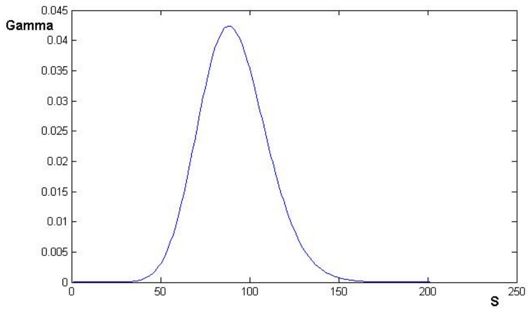


Figure 9: Black-Scholes Gamma of a call option with strike $K = 100$ as a function of the underlying level $S$

Due to put-call parity, the put Gamma is the same.