B8.3 Week 6 summary 2020

Gamma hedging

We can reduce the discretization error of the hedge over the first time step by **Gamma-hedging**.

We cancel the current (instantaneous) Gamma of our option V by opening a position in another option V^1. Typically, we hedge an exotic option with underlying and a vanilla call or put. The current value of the resulting portfolio is given by

$$-V_t + \Delta^1 V_t^1 + \Delta S_t + \gamma B_t = 0,$$

due to self-financing. We would like it to stay close to zero at time $t+\delta t$.

The above portfolio is **Gamma-neutral** if

$$\frac{\partial^2}{\partial S^2} V_t - \Delta^1 \frac{\partial^2}{\partial S^2} V_t^1 = 0.$$

So

$$\Delta^1 = \frac{\frac{\partial^2}{\partial S^2} V_t}{\frac{\partial^2}{\partial S^2} V_t^1} = \frac{\Gamma_t}{\Gamma_t^1}.$$

Thus, we obtain a new option which is a linear combination of V and V^1. We then Delta-hedge this new option:

$$\Delta = \frac{\partial}{\partial S} V_t - \Delta^1 \frac{\partial}{\partial S} V_t^1$$

So, strictly speaking, Gamma-hedging is a way to construct an instantaneous Gamma-neutral portfolio, which is then Delta-hedged.

It is "instantaneous" because it is no longer Gamma-neutral at time $t + \delta t$. And, if we want to repeat the procedure over the next time interval, we need to change Δ^1.

Example 1 *Let V be the price of a 91-day call with strike 100 and V^1 be the price of a 181-day call with strike 105.*

Assume that $S_t = 100$, $r = 0.05$, $\sigma = 0.3$.

Then, we have:
$V_t = 6.5583$, $\frac{\partial}{\partial S} V_t = 0.56202$, $\frac{\partial^2}{\partial S^2} V_t = 0.02631$,
and
$V_t^1 = 7.3295$, $\frac{\partial}{\partial S} V_t^1 = 0.49569$, $\frac{\partial^2}{\partial S^2} V_t^1 = 0.01888$.

Applying the above formulas, we obtain:

$\Delta^1 = 1.394$, $\Delta = -0.1290$, $\Delta^1 V_t^1 + \Delta S_t - V_t = -9.241$.

If the underlying changes to $S_{t+\delta t} = 101$*, then* $V_{t+\delta t} = 7.094$*,* $V_{t+\delta t}^1 = 7.8052$*,* $\Delta^1 V_t^1 + \Delta S_t - V_t = -9.243$*.*

If the underlying changes to $S_{t+\delta t} = 105$*, then* $V_{t+\delta t} = 9.645$*,* $V_{t+\delta t}^1 = 10.009$*,* $\Delta^1 V_t^1 + \Delta S_t - V_t = -9.237$*.*

The increments from the bank account are, as usual negligible provided δt *is small. Thus, we conclude that the hedging error (which is due to discretization), relative to the increment of the underlying asset, is of the order* 10^{-3}.

Robustness of the Black-Scholes formula

We now see a remarkable robustness property of BS-style hedging. We know that the stock price dynamics in the BS model are almost certainly wrong, but this does not necessarilty imply that we cannot use a delta-hedging rule based on the BS formula to achieve a successful hedge, even in the face of severe model error, as the following argument shows.

Suppose the true price process of a stock is

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t,$$

where $(\mu_t, \sigma_t)_{t\geq 0}$ are processes adapted to a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. The market is not necessarily complete, so the filtration $\mathbb{F}$ can be larger than filtration generated by the BM W.

Suppose a trader sell an option (say, a call with some maturity T) at time zero using an IV of σ_0. That is, the option is sold for $v(0, S_0)$ where $v(t, x)$ solves the BS PDE with volatility σ_0:

$$v_t(t,x) + rxv_x(t,x) + \frac{1}{2}\sigma_0^2 x^2 v_{xx}(t,x) - rv(t,x) = 0. \tag{1}$$

The trader uses the proceeds of the option sale to form a hedge portfolio with initial value $X_0 = v(0, S_0)$, and then uses the hedge $H_t = v_x(t, S_t)$ (so that $X_t - H_t S_t$ is in cash) at $t \in [0, T]$.

Define $R_t := X_t - v(t, S_t)$, as the "tracking error" (or residual risk). Using the Itô formula and the PDE satisfied by $v(t, x)$, we have (exercise!)

$$d(\mathrm{e}^{-rt} R_t) = \frac{1}{2}\mathrm{e}^{-rt} S_t^2 v_{xx}(t, S_t)(\sigma_0^2 - \sigma_t^2)dt.$$

We conclude that since $v_{xx}(t, S_t) \geq 0$ (for both a call and a put) we have $R_T \geq 0$ a.s. if $\sigma_0 \geq \sigma_t$ for all $t \in [0, T]$. In other words, the hedging strategy makes a profit with probability 1 if the implied volatility σ_0 is high enough. In this sense, successful hedging is entirely a matter of good volatility estimation.

This is a crucial result, as it shows that successful hedging is quite possible even under significant model error. Without some robustness property of this kind, it is hard to imagine that the derivatives industry could exist at all.

Volatility and Vega.

Volatility σ is the only parameter in the Black-Scholes model that is **not directly observed** in the market.

It is, therefore, important to be able to evaluate the dependence of option price on volatility.

The corresponding sensitivity is called **Vega**:

$$\nu = \frac{\partial}{\partial\sigma} V$$

In the BS model, σ is constant, so hedging with respect to changes in σ doesn't make sense.

However, one can ask: *what if my estimate of σ is wrong*?

To estimate how far off, in this case, the computed option price is from the "true" price, we need to find Vega.

For a call option, we have

$$\nu^{call} = \sqrt{\frac{T-t}{2\pi}} S e^{-\frac{d_+^2}{2}}.$$

And it tends to zero as $t \to T$, since the payoff is independent of σ.

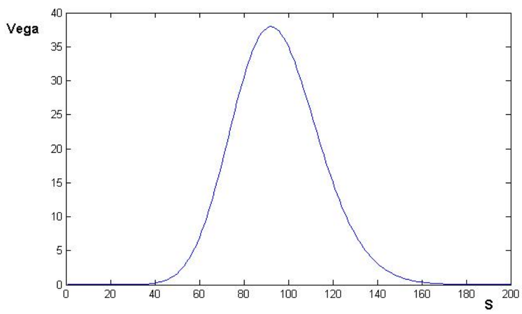

Figure 1: Black-Scholes Vega of a call option with strike $K = 100$ as a function of the underlying level S

It is the same for a put, due to put-call parity.

Vega-hedging can be defined in the same way as Gamma-hedging, however, its purpose is different:

rather than reducing the discretization error, it is meant to reduce the **model error** – a misspecification of σ.

Given an additional derivative with price V^1, the Vega-hedge of a short position in the original option prescribes to hold Δ^1 units of V^1 and Δ units of S.

In order to make the portfolio instantaneously **Vega-neutral**, we need

$$-\frac{\partial}{\partial\sigma}V_t + \Delta^1\frac{\partial}{\partial\sigma}V_t^1 = 0.$$

Therefore,

$$\Delta^1 = \frac{\frac{\partial}{\partial\sigma}V_t}{\frac{\partial}{\partial\sigma}V_t^1} = \frac{\nu_t}{\nu_t^1}$$

As before, Δ is determined as the corresponding S-derivative of the portfolio of options $V - \Delta^1 V^1$, assuming Δ^1 is fixed:

$$\Delta = \frac{\partial}{\partial S}V_t - \Delta^1\frac{\partial}{\partial S}V_t^1$$

Of course, in order to keep the portfolio Vega-neutral at the next moment in time $t+\delta t$, the value of Δ^1 (as well as Δ) will need to be changed at that time.

Example 2 *Consider a call option with* $T = 90/252$*,* $r = 0.05$*,* $\sigma = 0.2$*,* $K = 100$*,* $S = 100$*. Then* $V = 4.5922$*,* $\nu = 19.654$*.*

The additional derivative is a call with $T = 180/252$ *and* $K = 105$*.*

Then $V^1 = 4.9925$*,* $\nu^1 = 26.727$*,* $\Delta^1 = \nu/\nu^1 = 0.735$*, and the total value invested in the options according to the Vega-hedge is* $\Delta^1 V^1 - V = -1.4026$*.*

Assuming the value of the underlying and time do not change, if the "true" σ *turns out to be* 0.22*, then* $V = 4.9925$*,* $V^1 = 4.8748$ *and the total value invested in options by the Vega-hedge is* $\Delta^1 V^1 - V = 1.4095$*.*

We can see that, even the error in the option price, due to misspecification of volatility, is around 0.4*, the Vega-neutral portfolio is only mispriced by less than* $7 \cdot 10^{-3}$*.*

Remark 1 *Vega hedging makes sense if one believes that the "true" volatility is constant, but we may be mistaken about its true value.*

However, sometimes, you may see a description of Vega-hedging as "hedging the non-constant volatility". It is good to remember that Vega-hedging ***is not the right hedging strategy in the presence of dynamically changing volatility!*** *In fact, if the volatility is believed to be changing dynamically, then the more complicated* ***stochastic volatility models*** *have to be used to compute the right hedge.*

Using a ***constant volatility model****, such as the Black-Scholes, to design a* ***hedge against stochastic volatility*** *is, clearly,* ***self-contradictory****. It may sometimes be used in practice, if other options are too hard to implement, however, then, one has to be very careful and make sure that the side effects of such hedging do not overweigh the positive impact.*

Vega and Gamma.

In fact, there is a **universal relation between Vega and Gamma** which holds for **all European options**, because their prices are **functions of time and the value of the underlying** and these functions **satisfy the BSPDE**.

Consider the BSPDE

$$\frac{\partial}{\partial t}V + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2}V + rS\frac{\partial}{\partial S}V - rV = 0$$

and differentiate it with respect to σ.

For example,

$$\frac{\partial}{\partial \sigma}V = \nu,$$

$$\frac{\partial^2}{\partial \sigma \partial t}V = \frac{\partial}{\partial t}\nu$$

and so on.

As a result, we obtain

$$\mathcal{L}_{BS}\nu = \frac{\partial}{\partial t}\nu + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2}\nu + rS\frac{\partial}{\partial S}\nu - r\nu = -\sigma S^2 \frac{\partial^2}{\partial S^2}V = -\sigma S^2 \Gamma, \quad (2)$$

with $\nu(S,T) = 0$.

In PDE language, the above equation means that "$-\sigma S^2\Gamma$" is a *source* for ν.

Notice also that $\sigma S^2 \Gamma = \sigma S^2 \frac{\partial^2}{\partial S^2}V$ satisfies the BSPDE (recall homework exercise).

Therefore, it is easy to check that

$$\nu(S,t) = (T-t)\sigma S^2 \Gamma(S,t)$$

satisfies the PDE (2) with the terminal condition $\nu(S,T) = 0$.

Indeed, we have

$$\mathcal{L}_{BS}\left[(T-t)\sigma S^2 \Gamma(S,t)\right] = -\sigma S^2 \Gamma + (T-t)\mathcal{L}_{BS}\left[\sigma S^2 \Gamma(S,t)\right] = -\sigma S^2 \Gamma$$

This is a useful trick, and a good example of how the **PDE techniques** may help in establishing certain **non-trivial** relations between various quantities in mathematical finance.

Other Greeks

There are also **higher order sensitivities**, such as **Vanna**.

Vanna is the sensitivity of Delta with respect to changes in volatility σ.

This is a measure of model dependence of the Delta-hedging strategy itself.

Theta Θ is sensitivity with respect to t, and measures the maturity sensitivity of our portfolio. **Rho** ρ is sensitivity with respect to the interest rate r, and measures the potential impact of interest rate changes on the value of the portfolio (more important over the long-term).

Given enough trading instruments (assets, options), we can cancel the higher order sensitivities as well.

However, decreasing the risk associated with a wrong choice of parameters, we can increase the more general model risk: that our family of models (the BS models, parameterized by r and σ) is wrong itself!

Therefore, one should find an optimal trade-off, and **shouldn't go too far with "matching the Greeks".**

> *Timeo Danaos et dona ferentes* (I fear the Greeks, even those bearing gifts) —Virgil, Aeneid

Some properties of solutions of the Black-Scholes equation

The Black-Scholes equation is

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-y)\, S\, \frac{\partial V}{\partial S} - r\, V = 0, \quad S > 0,\ t < T \qquad (3)$$

and it has the following properties:

1. it is linear;

2. it is solved *backwards* in time, for $t < T$;

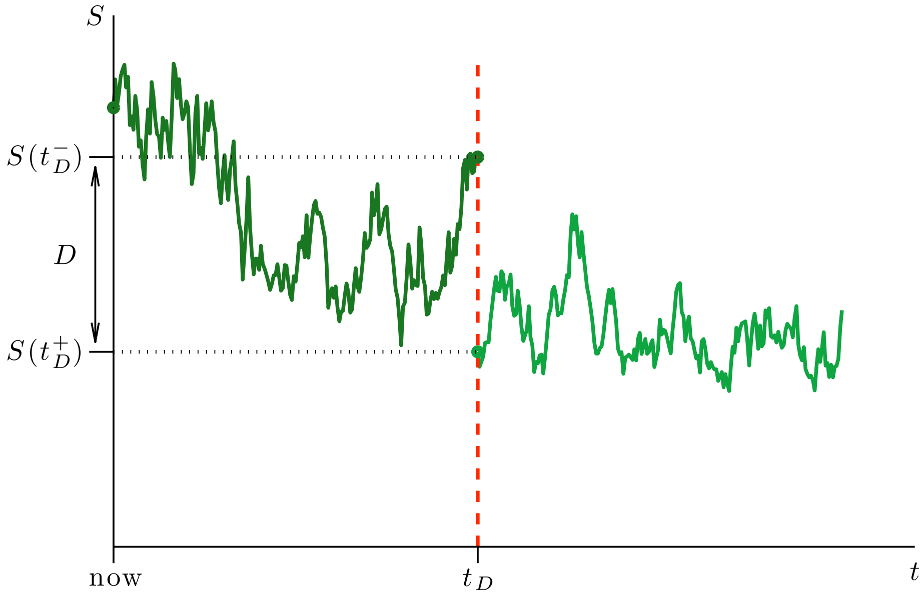

Figure 2: Jump in share price across a discrete dividend date.

3. if $V(S,t)$ is a solution so too is $V(\lambda S, t)$ for any $\lambda > 0$;
4. $V(S,t)$ depends on t and T only through the combination $T-t$;
5. if $V(S,t)$ is a solution so too is $S\big(\partial V/\partial S\big)$ (and, by induction, so too are $S^n\big(\partial^n V/\partial S^n\big)$ for $n = 2, 3, \ldots$);
6. if $V(S,t)$ is a solution so too is

$$\hat{V}(S,t) = (S/A)^{2\alpha}\, V\big(B^2/S, t\big), \quad 2\alpha = 1 - 2(r-y)/\sigma^2,$$

for any constants $A > 0$, $B^2 > 0$.

Discrete dividends

Suppose that a share pays a deterministic dividend D at time t_D. If both D and t_D are known in advance we must have

$$S_{t_D^-} = S_{t_D^+} + D \quad \Longleftrightarrow \quad S_{t_D^+} = S_{t_D^-} - D$$

otherwise there is an arbitrage opportunity. If we have an option on this share then we *don't* get the dividend and so we must have the jump condition

$$V\big(S_{t_D^-}, t_D^-\big) = V\big(S_{t_D^+}, t_D^+\big) = V\big(S_{t_D^-} - D, t_D^+\big).$$

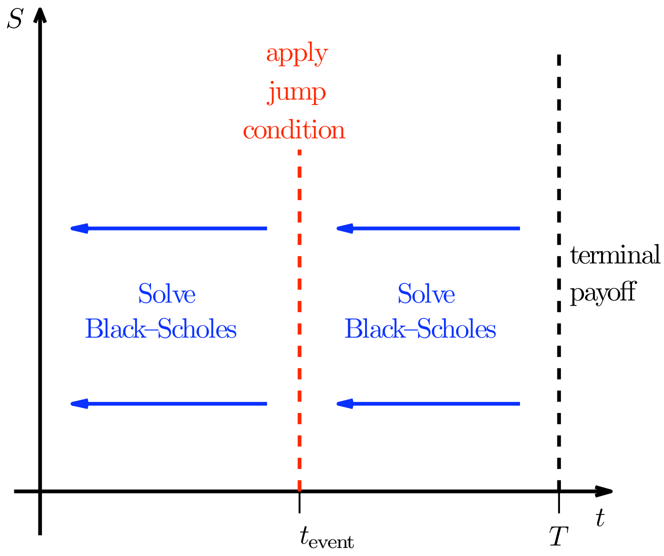

Figure 3: General strategy for dealing with a discrete-time event.

As this is true for any $S_{t_d^-}$ and we solve the Black-Scholes equation backwards in time, we generally write this *jump condition* as

$$V(S, t_D^-) = V(S - D, t_{D^+}). \tag{4}$$

The strategy is to solve the Black-Scholes equation back from expiry, T, until the dividend date t_D^+, then apply (4) to find $V(S, t_D^-)$ and then solve the Black-Scholes equation backwards from t_D^- to the present time, using $V(S, t_D^-)$ as a "payoff" at t_D^-.

Note that D can be a function of S and t. Indeed, if we want the share price to remain positive, it must be. Modelling discrete dividend payments for a share price that follows geometric Brownian motion is problematic to this day.

Discrete dividend yields

If we assume a discrete dividend of the form

$$D = d_y \, S_{t_d^-},$$

where the discrete dividend yield $d_y < 1$, i.e., the dividend is proportional to the share price immediately before the dividend is paid then we find that

$$S_{t_d^-} = S_{t_d^+} + d_y \, S_{t_d^-} \quad \Longleftrightarrow \quad S_{t_d^+} = (1 - d_y) \, S_{t_d^-}$$

Jump condition across a discrete dividend yield date

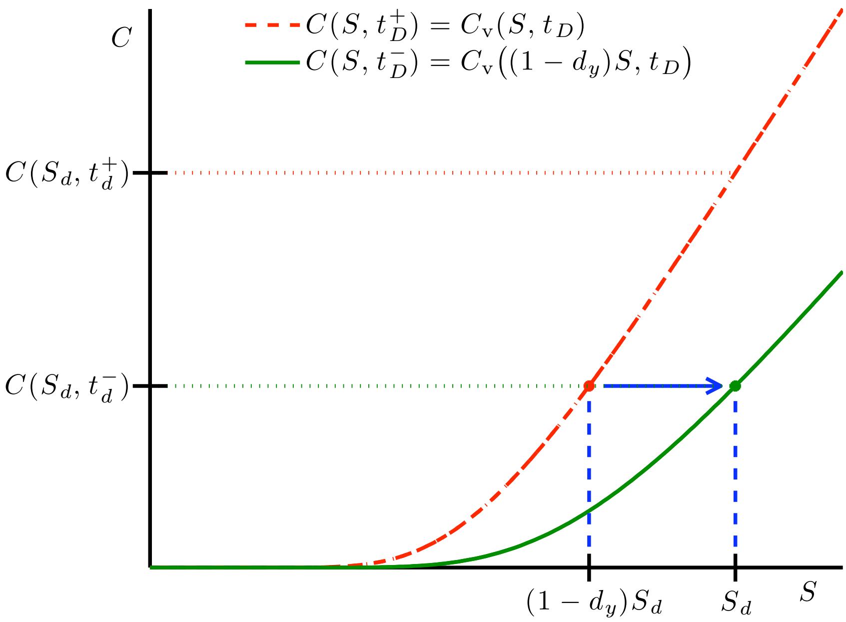

Figure 4: Jump condition for a call option on a share that pays a discrete dividend yield.

and the jump condition for the option becomes

$$V(S,t_d^-) = V\big((1-d_y)S,t_d^+\big). \tag{5}$$

We can then use the fact that if $V(S,t)$ is a solution of the Black-Scholes equation then so too is $V(\lambda S,t)$, with $\lambda = (1-d_y)$ in this case, to see that the solution for $t < t_d$ is simply

$$V\big((1-d_y)S,t\big),$$

as it is a solution of the Black-Scholes equation and obviously satisfies the "payoff" condition at t_d^-.

A call option with one discrete dividend yield

Let $C_v(S,t)$ be the price function for a vanilla call, i.e.,

$$\begin{aligned} C_v(S,t) &= S\,\mathrm{N}(d_+) - K\,e^{-r(T-t)}\,\mathrm{N}(d_-), \\ d_\pm &= \frac{\log(S/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sqrt{\sigma^2(T-t)}.} \end{aligned}$$

Let the share pay a discrete dividend yield of d_y at time $0 < t_d < T$ and let $C(S,t)$ be the price function for a call written on this share. Then for

$t_d < t < T$ we have

$$C(S,t) = C_{\mathrm{v}}(S,t).$$

Across the dividend date t_d, we apply (5) to get

$$C(S,t_d^-) = C_{\mathrm{v}}\big((1-d_y)S, t_d\big)$$

and then note that as $1-d_y > 0$ is a constant, the function $C_{\mathrm{v}}\big((1-d_y)S, t_d\big)$ is itself a solution of the Black-Scholes equation and so for all $t < t_d$ we have

$$C(S,t) = C_{\mathrm{v}}\big((1-d_y)S, t\big).$$

The same reasoning shows that if there are n discrete dividend yields at times

$$t < t_1 < t_2 < \cdots < t_n < T$$

between now and expiry with dividend yields

$$d_1,\ d_2,\ \ldots\ ,\ d_n,$$

where each $d_k < 1$, then

$$C(S,t) = C\big(\alpha_n S, t\big), \quad \text{where} \quad \alpha_n = \prod_{k=1}^{n}(1-d_k).$$

Clearly this result generalises to any European option, regardless of the its payoff.

American options

An American option is an option which can be exercised at any time between being initiated and expiring (inclusive). It follows that

- It can not be less valuable than the payoff $P_{\mathrm{o}}(S_t, t)$, which may depend on t because the option can be exercised at any time $0 \le t \le T$. If it were, the arbitrage is to buy the option and immediately exercise it to receive the payoff (which is greater than the price).
- It can't be worth less than an otherwise equivalent European option. If it were, the arbitrage is to buy the American option, write the European option and put the positive profit in the bank. Then hold the American option until expiry (this may not be optimal, but if you own the American option you are free to do it) at which point it equals the European option and so you are perfectly covered.

It is easy to see that if $r > 0$ then for a European put we have

$$\lim_{S\to 0} P(S,t) = K\, e^{-r(T-t)} < K.$$

Since the European put price is differentiable, it is also continuous and so this shows that prior to expiry a European put is less valuable than the payoff for small enough S. As an American put can't be less valuable than the payoff, the values of American and European puts must be different. As they both have the same payoff, $(K - S)^+$, the American put can't satisfy the Black-Scholes equation for all $S > 0$.