

B8.3 Week 8 summary 2020

Down and out barrier call options

A down and out barrier call option becomes worthless (colloquially referred to as “knocking out”) if the share price falls to or below a barrier level, $B > 0$, at any time during the option’s life. In these notes, we take B to be a constant. If $S_t > B$ for all $t \in [0, T]$ then it has payoff $(S_T - K)^+$. The pricing problem, assuming that the option has not already knocked out, is

$$\frac{\partial C_{\text{do}}}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_{\text{do}}}{\partial S^2} + (r - y) S \frac{\partial C_{\text{do}}}{\partial S} - r C_{\text{do}} = 0, \quad S > B, \quad t < T,$$
$$C_{\text{do}}(S, T) = (S - K)^+, \quad S > B, \quad C_{\text{do}}(B, t) = 0, \quad t \leq T.$$

The case $0 < B \leq K$

The trick here is to recall that if $V(S, t)$ is a solution of the Black-Scholes equation then so too is

$$\hat{V}(S, t) = (S/B)^{2\alpha} V(B^2/S, t),$$

where $2\alpha = 1 - 2(r - y)/\sigma^2$, and that

$$\hat{V}(B, t) = (B/B)^{2\alpha} V(B^2/B, t) = V(B, t).$$

So if $C_{\text{bs}}(S, t)$ is the price of a vanilla call option we see that

$$C_{\text{do}}(S, t; B) = C_{\text{bs}}(S, t) - (S/B)^{2\alpha} C_{\text{bs}}(B^2/S, t)$$

is a solution of the Black-Scholes equation which satisfies $C_{\text{do}}(B, t; B) = 0$. Then we notice that as $B < S$ and $B \leq K$ we have $B^2/S < B \leq K$ so that

$$C_{\text{bs}}(B^2/S, T) = (B^2/S - K)^+ = 0$$

which shows that for $S > B$

$$C_{\text{do}}(S, T; B) = C_{\text{bs}}(S, T) = (S - K)^+.$$

Thus $C_{\text{do}}(S, t; B)$ satisfies the pricing problem and is the Black-Scholes value of the barrier option.

The case $B > K > 0$

In this case the trick above fails because we find that $C(B^2/S, T) \neq 0$ for all $S > B$. The way to deal with it is to truncate the payoff of the call so that it becomes equal to zero if $S \leq B$ but remains unchanged if $S > B$, i.e., replacing the vanilla call above with an option whose payoff is

$$V(S, T) = \begin{cases} 0 & \text{if } 0 < S \leq B, \\ S - K & \text{if } S > B > K. \end{cases}$$

This payoff can be achieved by using a vanilla call with strike B plus $(B - K)$ digital calls, also with strike B . So instead of using the vanilla call price as above, we work with

$$V(S, t; B) = C_{bs}(S, t; K = B) + (B - K) C_d(S, t; K = B).$$

The Black-Scholes price function is given by

$$C_{do}(S, t; B) = V(S, t; B) - (S/B)^{2\alpha} V(B^2/S, t)$$

since this satisfies the Black-Scholes equation and boundary condition at $S = B$ and if $S > B$ then $B^2/S < B$ and so $V(B^2/S, T) = 0$, giving us the correct payoff at T .

The down-and-in barrier call

This option remains worthless if the share price does *not* fall *below* the barrier $B > 0$ during the life of the option. If at some point during the life of the option we have $S_t < B$ then the option turns into a vanilla call with payoff $(S_T - K)^+$; this is often referred to as “knocking in”. If we hold both a down-and-out and a down-and-in call option then we are guaranteed the payoff $(S_T - K)^+$ and so there is a down-and-in / down-and-out parity relation,

$$C_{do}(S, t; B) + C_{di}(S, t; B) = C_{bs}(S, t)$$

and hence

$$C_{di}(S, t; B) = C_{bs}(S, t) - C_{do}(S, t; B).$$

In the case that $B < K$ this simplifies to

$$C_{di}(S, t; B) = (S/B)^{2\alpha} C_{bs}(B^2/S, t).$$

Note that these formula for C_{di} are only valid if $S \geq B$ for all time up to the present. As soon as $S < B$ the option turns into a vanilla call and remains so until expiry.

Asian options

Asian options are options which depend on the average share price over the life of the option. In practice, it is usually the arithmetic average which we can define using the running sum of the share price

$$R_t = \int_0^t S_u du, \quad A_T = R_T/T = \frac{1}{T} \int_0^T S_u du,$$

where A_T is the average price at T . The option's price is a function of S_t , R_t and t , $V_t = V(S_t, R_t, t)$ for some function $V(S, R, t)$. If we note that

$$dR_t = S_t dt$$

and assume that $dW_t^2 = dt$ (which really means $d[W]_t = dt$) and perform a formal Taylor series expansion then we find

$$\begin{aligned} dV_t &= \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \frac{\partial V}{\partial R} dR_t + \frac{\partial V}{\partial S} dS_t \\ &= \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + S_t \frac{\partial V}{\partial R} \right) dt + \frac{\partial V}{\partial S} dS_t, \end{aligned}$$

where all partial derivatives are evaluated at (S_t, R_t, t) . Applying the usual hedging (or self-financing replication) argument(s) shows that to eliminate risk we must hold

$$\Delta_t = \Delta(S_t, R_t, t) = \frac{\partial V}{\partial S}(S_t, R_t, t)$$

shares at time t and that the pricing equation is

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - y) S \frac{\partial V}{\partial S} + S \frac{\partial V}{\partial R} - r V = 0. \quad (1)$$

This holds for all $S > 0$, $t < T$ and $R > 0$; as $S_u > 0$, $R_t = \int_0^t S_u du$ can take only positive values.

If the option is what is known as a floating-strike asian call, where the average plays the role of the strike, so the payoff is

$$V(S, R, T) = (S - R/T)^+,$$

then we can simplify the problem by working the variables

$$x = R/S, \quad V(S, R, t) = S u(x, t),$$

i.e., by pricing relative to the share price rather than a unit of currency (in finance this is usually called *a change of numeraire*). We find that

$$\begin{aligned} \frac{\partial V}{\partial t} &= S \frac{\partial u}{\partial t}, \quad S \frac{\partial V}{\partial R} = S \frac{\partial u}{\partial x}, \\ S \frac{\partial V}{\partial S} &= S \left(u - x \frac{\partial u}{\partial x} \right), \quad S^2 \frac{\partial^2 V}{\partial S^2} = S x^2 \frac{\partial^2 u}{\partial x^2}. \end{aligned}$$

Substituting these into the pricing equation (1) gives

$$\frac{\partial u}{\partial t} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} + (1 + (y - r)x) \frac{\partial u}{\partial x} - y u = 0. \quad (2)$$

At expiry, $t = T$, we have $V(S, R, T) = (S - R/T)^+ = S u(x, T)$ and so

$$u(x, T) = (1 - x/T)^+. \quad (3)$$

The Feynman–Kac formula shows that the solution of problem (2)–(3) can be expressed as

$$u(x, t) = e^{-y(T-t)} \mathbb{E} \left[\left(1 - x_T/T \right)^+ \mid x_t = x \right],$$

where x_τ evolves as

$$dx_\tau = \left(1 + (y - r) x_\tau \right) d\tau + \sigma x_\tau dW_\tau,$$

for $\tau > t$, with $x_t = x$.