Further Quantum Theory: Problem Sheet 4

Hilary Term 2020

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4.1 Anharmonic oscillator WKB

Consider a (possibly anharmonic) oscillator in one dimension, with Hamiltonian

$$H = \tfrac{1}{2m}P^2 + \left(\tfrac{1}{2}m\omega^2 X^2\right)^k \ .$$

Derive the (integral) WKB quantization condition that implicitly determines the quantum energy levels of this system in the semi-classical approximation. By rescaling the integrand, show that the energy levels are given by

$$E_n = \left(\frac{\pi^{1/2}\Gamma(\frac{3k+1}{2k})}{2\Gamma(\frac{2k+1}{2k})}\hbar\omega(n+\tfrac{1}{2})\right)^{\frac{2k}{k+1}} \ ,$$

where you will need to utilize the integral formula

$$\int_{-1}^{1}\sqrt{1-x^{2k}} = \frac{\pi^{1/2}\Gamma(\frac{2k+1}{2k})}{\Gamma(\frac{3k+1}{2k})} \ .$$

Check the case $k=1$, which corresponds to the simple harmonic oscillator.

4.2 Spherical WKB approximation

Find the WKB approximation for stationary, spherically symmetric, bound states (*i.e.*, states with angular momentum $l=0$ and $E<0$) of an electron of mass m in a Coulomb potential, with the usual Hydrogen-like Hamiltonian,

$$E = \frac{P^2}{2m} - \frac{Z{q_e}^2}{r} \ .$$

(i) Imposing that the wave function is bounded at $r=0$ will lead to a Bohr-Sommerfeld-like quantization condition. Compare this to the exact answer for the Hydrogen energy levels.

(ii) Now analyze the case with angular momentum $l \neq 0$. Here there is both an inner and an outer turning point. Remember that the principle quantum number in the Hydrogen atom is given by $n = k+l+1$, where k is the number of zeroes of the radial wavefunction. Compare your answer to the exact energy levels.

(iii) The *Langer correction* to the WKB analysis of the Hydrogen atom proceeds by replacing $l(l+1)$ in the centrifugal potential term with $(l+\frac{1}{2})^2$. This includes the case $l=0$. Check that the WKB results improve dramatically upon including the Langer correction.

[*You will also need to be able to perform the integral*

$$\int_a^b \frac{\sqrt{(r-a)(b-r)}}{r} = \frac{\pi}{2}\left(a^{1/2}-b^{1/2}\right)^2 \ , \qquad a<b \ .$$

I've changed to denoting the charge of the electron as q_e *to avoid any ambiguity related to Euler's constant.*]

4.3 Quantum tunnelling with WKB

A particle is incident from the left upon a potential barrier of the form

$$V(x) = \begin{cases} 0 \ , & x<-a \ , \\ a^2-x^2 \ , & -a<x<a \ , \\ 0 \ , & x>a \ . \end{cases}$$

Set up a WKB approximation for the (non-normalizable) stationary wave function describing this system for both cases $E > a^2$ and $E < a^2$. For the latter case, determine the *transmission amplitude* through, and the *reflection amplitude* off of, the classical barrier as a function of the energy of the incident particle.

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