# B7.3 Further quantum theory lecture 3

## Composite systems

Suppose two (non-interacting) quantum systems are to be considered together.

$$\left.\begin{array}{l}\text{System } 1 \longleftrightarrow \mathcal{H}_1 \\ \text{System } 2 \longleftrightarrow \mathcal{H}_2\end{array}\right\} \text{composite system} \longleftrightarrow \mathcal{H}_J$$

How do we describe this system?

- For any $|\psi_1\rangle \in \mathcal{H}_1$, $|\psi_2\rangle \in \mathcal{H}_2$ we should have a state $|\psi_1\rangle \otimes |\psi_2\rangle \equiv |\psi_1 \otimes \psi_2\rangle$
- Take vector space $F(\mathcal{H}_1, \mathcal{H}_2)$ given by $\mathbb{C}$-linear combinations of these.
  (this is the "free vector space over the set $\mathcal{H}_1 \times \mathcal{H}_2$").
- This overcounts, we need to make some identifications.
  - $|(\psi_1 + \phi_1) \otimes \psi_2\rangle \sim |\psi_1 \otimes \psi_2\rangle + |\phi_1 \otimes \psi_2\rangle$
  - $|\psi_1 \otimes (\psi_2 + \phi_2)\rangle \sim |\psi_1 \otimes \psi_2\rangle + |\psi_1 \otimes \phi_2\rangle$
  - $|\alpha\psi_1 \otimes \psi_2\rangle \sim \alpha|\psi_1 \otimes \psi_2\rangle \sim |\psi_1 \otimes \alpha\psi_2\rangle \quad \forall \alpha \in \mathbb{C}$
  - $\mathcal{H}_1 \otimes \mathcal{H}_2 \cong F(\mathcal{H}_1 \times \mathcal{H}_2)/\sim$
- Inner product inherited from $\mathcal{H}_1$ & $\mathcal{H}_2$:
  - $\langle \psi_1 \otimes \psi_2 | \phi_1 \otimes \phi_2 \rangle = \langle \psi_1 | \phi_1 \rangle \langle \psi_2 | \phi_2 \rangle$
  - Extend to general elements by (conjugate) linearity.

These conditions define the tensor product $\mathcal{H}_J \cong \mathcal{H}_1 \otimes \mathcal{H}_2$.[†]

[†] For Hilbert space tensor product, need to further take "completion" with respect to inner product. This won't play any role for us.

Can also define tensor product in terms of orthonormal bases:

Let $|\alpha_{i\in I}\rangle$ be an orthonormal basis for $\mathcal{H}_1$,
$|\beta_{j\in J}\rangle$ be an orthonormal basis for $\mathcal{H}_2$

Then $|\alpha_i \otimes \beta_j\rangle$ form an orthonormal basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$.

This is equivalent to the previous, abstract definition, so the apparent dependance on a choice of bases for $\mathcal{H}_{1,2}$ is immaterial.

when finite-dimensional

Can easily check now that $\dim(\mathcal{H}_1 \otimes \mathcal{H}_2) = (\dim \mathcal{H}_1) \times (\dim \mathcal{H}_2)$.

Classically, one expects $\boxed{\text{states of system 1}} \overset{\text{Cartesian Product}}{\times} \boxed{\text{states of system 2}} = \boxed{\text{states of composite}}$, in which case dimensions <u>add</u>.

It's the principle of superposition (ability to take arbitrary linear combinations of states) that gives multiplicative behaviour.

Definition: a state $|v \otimes w\rangle$ is a decomposable tensor (also called a pure tensor). (Note this is a basis-independent property.)

A state that is not decomposable is said to be an entangled state.

The space of decomposable tensors in $\mathcal{H}_1 \otimes \mathcal{H}_2$ is a (non-linear) subspace of dimension $n+m-1$.

(The space of decomposable states in $\mathbb{P}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is a subspace of the form $\mathbb{P}(\mathcal{H}_1) \times \mathbb{P}(\mathcal{H}_2)$).

$$\overbrace{\mathbb{P}(\mathcal{H}_1)}^{(n-1)\text{-dim'l}} \times \overbrace{\mathbb{P}(\mathcal{H}_2)}^{(n-1)\text{-dim'l}} \xhookrightarrow[\text{Segre embedding}]{} \overbrace{\mathbb{P}(\mathcal{H}_1 \otimes \mathcal{H}_2)}^{(nm-1)\text{-dim'l}}$$

Since $n+m-1 < n \times m$ for any $n,m \geq 2$, we see that generically states are entangled.

# Tensor products of qubits

The quantum system with $\mathcal{H} \cong \mathbb{C}^2$ is called a qubit (pronounced "cube-it").
Denote orthonormal basis vectors $|+\rangle$ and $|-\rangle$ (also see $|0\rangle$ & $|1\rangle$, or $|\uparrow\rangle$ and $|\downarrow\rangle$).

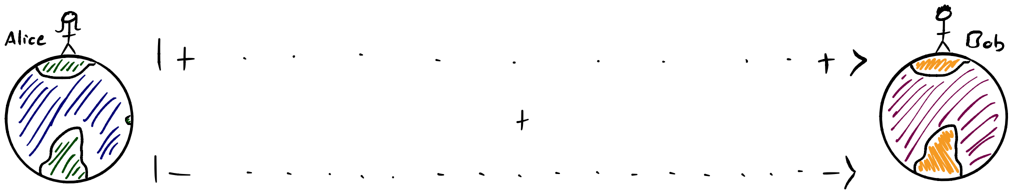
The two qubit system has $\mathcal{H} \cong \mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4$ with basis $\{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$.

If $|\psi\rangle = a_{++}|++\rangle + a_{+-}|+-\rangle + a_{-+}|-+\rangle + a_{--}|--\rangle$, can show (exercise!) that $|\psi\rangle$ is entangled if and only if the determinant $\begin{vmatrix} a_{++} & a_{+-} \\ a_{-+} & a_{--} \end{vmatrix} \neq 0$.

## A bit on entanglement:

EPR state: $|EPR\rangle = \dfrac{|++\rangle + |--\rangle}{\sqrt{2}}$ (Einstein-Podolsky-Rosen)

Suppose the two qubit system is distributed between Alice, who holds the first qubit, and Bob who holds the second. Alice stays on earth while Bob flies to a distant solar system.




Now Alice performs a measurement to find out if her qubit is in the $|+\rangle$ or $|-\rangle$ state. (Corresponding to, e.g., the Hermitian operator $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.)

If she measures $|+\rangle$, then the state "collapses" to $|++\rangle$, at which point if Bob measures his qubit, he will find $|+\rangle$ with 100% certainty, and likewise for $|-\rangle$ if Alice's measurement finds $|-\rangle$.

EPR dubbed this "spooky action-at-a-distance", and took it as a sign that there must be additional "hidden variables" which dictate in advance which of the two possibilities is realized.

Notice that although after Alice's measurement, Bob's result is assured, he can't know what the (certain) result will be unless Alice makes contact by conventional means.

No instantaneous communication (cf., "No communication theorem").

(The model of hidden variables proposed by EPR was subsequently ruled out experimentally using something called Bell's inequalities.)

# Some other cases

## (1) Internal degrees of freedom

If $\mathcal{H}_1 \cong L^2(\mathbb{R})$ and $\mathcal{H}_2 \cong \mathbb{C}^n$ with orthonormal basis $|i\rangle_{i=1,\dots,n}$, then a general state takes the form

$$|\psi\rangle = \sum_{i=1}^{n} |\psi_i(x) \otimes i\rangle \qquad \left(\text{so } \mathcal{H}_1 \otimes \mathcal{H}_2 \cong (L^2(\mathbb{R}))^n\right)$$

$$= \sum_{i=1}^{n} \int_{-\infty}^{\infty} dx \; \psi_i(x) \, |x \otimes i\rangle$$

$$= \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \vdots \\ \psi_n(x) \end{pmatrix}$$

for physical states, individual component wave functions can be zero as long as some entry is nonzero

This is the kind of Hilbert space we will use to describe a particle with an internal label that takes one of $n$ values (in particular, in the case of spin).

## (2) Multiple particles

Now $\mathcal{H}_1 \cong L^2(\mathbb{R})$, $\mathcal{H}_2 \cong L^2(\mathbb{R}) \longrightarrow \mathcal{H}_1 \otimes \mathcal{H}_2 \cong L^2(\mathbb{R}^2)$

Can see this easily in terms of position eigenstates:

$$|\psi\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dy \; \psi(x,y) \, |x \otimes y\rangle$$

In this case, caveat involving completion with respect to inner product is necessary.

More generally, $(L^2(\mathbb{R}))^{\otimes n} \cong L^2(\mathbb{R}^n)$, so wave functions for multi-particle systems are as one would expect; squared-normalizable functions of all particle positions.