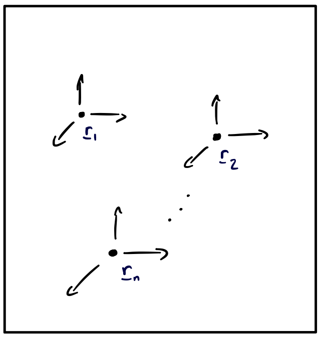
B7.3 Further quantum theory lecture 4

Previously we discussed composite systems and the tensor product. Now we focus on case of several particles moving in some number of spatial dimensions (say three).



$\longleftrightarrow$

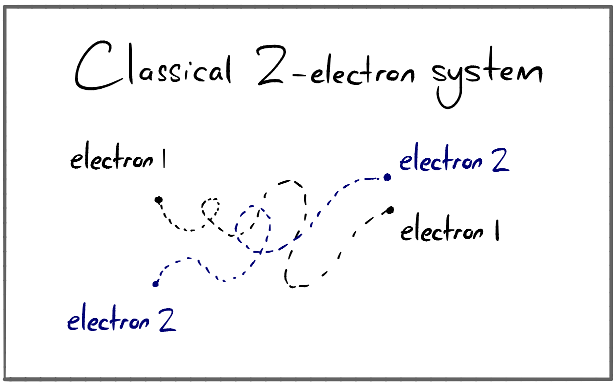
$$\Psi(r_1, r_2, \dots, r_n)$$

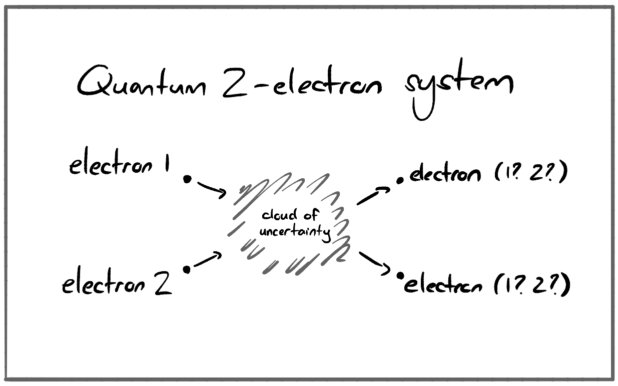
$$\int |\Psi(r_1, \dots, r_n)|^2 d^3 r_1 \cdots d^3 r_n = 1$$

$$|\Psi(r_1, \dots, r_n)|^2 = \text{prob. dens. for : } \begin{cases} \text{particle } 1 @ r_1 \\ \text{particle } 2 @ r_2 \\ \vdots \\ \text{particle } n @ r_n \end{cases}$$

We implicitly assumed that the constituent particles were distinguishable, even when the constituents themselves had identical descriptions.

Given the probabilistic nature of quantum mechanics, it isn't clear that this always makes sense.





Thus in quantum theory, we are led to posit that any two identical particles (e.g., some elementary particles, identical boundstates of elementary particles [e.g. Hydrogen atom ground state]) are physically indistinguishable.

Thus $|\Psi(r_1, r_2)|^2 d^3 r_1 d^3 r_2$ should be interpreted as the probability for finding some electron at r_1 and some electron at r_2, so...

$$|\Psi(r_1, r_2)|^2 = |\Psi(r_2, r_1)|^2$$

$$\text{thus } \Psi(r_1, r_2) = \lambda \Psi(r_2, r_1) \text{ for } |\lambda|^2 = 1$$

$$= \lambda^2 \Psi(r_1, r_2) \text{ so } \lambda = \pm 1.$$

We should consider the behaviour of $n>2$ identical particles under more general exchanges.

Let S_n denote the symmetric group on n-elements, and $\pi \in S_n$ some permutation. We should have a relation for the action of π on an n-particle wave-function:

$$\Psi(\underline{r}_1, \dots, \underline{r}_n) = \lambda(\pi)\Psi(\underline{r}_{\pi(1)}, \dots, \underline{r}_{\pi(2)})$$

For $\sigma \in S_n$ another permutation, we then have

$$\Psi(\underline{r}_1, \dots, \underline{r}_n) = \lambda(\sigma)\lambda(\pi)\Psi(\underline{r}_{\sigma\circ\pi(1)}, \dots, \underline{r}_{\sigma\circ\pi(n)})$$
$$= \lambda(\sigma\circ\pi)\Psi(\underline{r}_{\sigma\circ\pi(1)}, \dots, \underline{r}_{\sigma\circ\pi(n)})$$

So $\lambda : S_n \longrightarrow \mathbb{C}$, $\pi \longmapsto \lambda(\pi)$ is a one-dim'l representation of S_n. Observe that
$$\lambda(\sigma\pi\sigma^{-1}) = \lambda(\sigma)\lambda(\pi)\lambda(\sigma^{-1}) = \lambda(\sigma)\lambda(\pi)\lambda(\sigma)^{-1} = \lambda(\pi)$$

i.e., conjugate elements of S_n give same λ. Any transposition – say (rs) – is conjugate to transposition (12):

$$(rs) = (1r)(2s)(12)(2s)^{-1}(1r)^{-1}$$

so all transpositions give same λ. Also, S_n is generated by transpositions, with each permutation arising as products of either even or odd numbers of transpositions. Thus, we see:

option A: if $\Psi(\underline{r}_1, \underline{r}_2, \dots) = \Psi(\underline{r}_2, \underline{r}_1, \dots)$, then $\Psi(\underline{r}_1, \dots, \underline{r}_n) = \Psi(\underline{r}_{\pi(1)}, \dots, \underline{r}_{\pi(n)}) \quad \forall \pi \in S_n$

option B: if $\Psi(\underline{r}_1, \underline{r}_2, \dots) = -\Psi(\underline{r}_2, \underline{r}_1, \dots)$, then $\Psi(\underline{r}_1, \dots, \underline{r}_n) = \epsilon(\pi)\Psi(\underline{r}_{\pi(1)}, \dots, \underline{r}_{\pi(n)}) \quad \forall \pi \in S_n$

$$\text{where } \epsilon(\pi) = \begin{cases} 1 & \text{for } \pi \text{ even} \\ -1 & \text{for } \pi \text{ odd} \end{cases}$$

Physically, the choice of symmetry vs. anti-symmetry for wavefunctions should be a property of each "species" of fundamental particle. (Given two such particles and their exchange statistics, indistinguishability implies same for all such particles.)

<u>Def</u>: particles obeying option A are called "bosons" and are said to obey "Bose-Einstein statistics.

particles obeying option B are called "fermions" and are said to obey "Fermi-Dirac" statistics.

Indeed, these are what the world is made of:

Fermions: (elementary) electrons, neutrinos, quarks, (muons, taus)
(composites) protons, neutrons (odd # fermions)

Bosons: (elementary) photons, gluons, W & Z bosons, Higgs boson
(composites) pions, mesons, ... (even # fermions)

Thm. (Spin-Statistics): particles must be bosons if their spin is an integer, and they must be fermions if their spin is an integer $+\frac{1}{2}$.

We introduce some tools for working with/constructing explicit bosonic/fermionic wave functions.

Given $\Psi \in L^2(\mathbb{R}^{3n})$ we can define projection operators

$$Q_\lambda : L^2(\mathbb{R}^{3n}) \longrightarrow L^2(\mathbb{R}^{3n})$$

$$\Psi(\underline{r}_1, \dots, \underline{r}_n) \longmapsto \frac{1}{n!} \sum_{\pi \in S_n} \lambda(\pi) \Psi(\underline{r}_{\pi(1)}, \dots, \underline{r}_{\pi(n)})$$

For $\lambda(\pi) = 1$, $Q_\lambda \Psi$ is bosonic (or zero).
For $\lambda(\pi) = \epsilon(\pi)$, $Q_\lambda \Psi$ is fermionic (or zero).

Proof: $\lambda(\sigma)(Q_\lambda \Psi)(\underline{r}_{\sigma(1)}, \dots, \underline{r}_{\sigma(n)}) = \frac{1}{n!} \sum_{\pi \in S_n} \lambda(\pi\sigma) \Psi(\underline{r}_{\pi\sigma(1)}, \dots, \underline{r}_{\pi\sigma(n)})$

$$= \frac{1}{n!} \sum_{(\pi\sigma) \in S_n} \lambda(\pi\sigma) \Psi(\underline{r}_{\pi\sigma(1)}, \dots, \underline{r}_{\pi\sigma(n)})$$

$$= Q_\lambda \Psi(\underline{r}_1, \dots, \underline{r}_n) \qquad \blacksquare$$

Note also that $Q_\lambda(Q_\lambda \Psi) = \sum_{\pi \in S_n} \sum_{\sigma \in S_n} \frac{\lambda(\pi)\lambda(\sigma)}{(n!)^2} \Psi(\underline{r}_{\pi\sigma(1)}, \dots, \underline{r}_{\pi\sigma(n)})$ let $\pi = \rho\sigma^{-1}$

$$= \sum_{\rho \in S_n} \sum_{\sigma \in S_n} \frac{\overbrace{\lambda(\rho\sigma^{-1})\lambda(\sigma)}^{=\lambda(\rho)}}{(n!)^2} \Psi(\underline{r}_{\rho(1)}, \dots, \underline{r}_{\rho(n)})$$ sum over σ now gives $n!$

$$= \sum_{\sigma \in S_n} \frac{\lambda(\sigma)}{n!} \Psi(\underline{r}_{\sigma(1)}, \dots, \underline{r}_{\sigma(n)})$$

so indeed this is a projection onto bosonic/fermionic wave functions.

Simple case: $(Q_1 \Psi)(\underline{r}_1, \underline{r}_2) = \dfrac{\Psi(\underline{r}_1, \underline{r}_2) + \Psi(\underline{r}_2, \underline{r}_1)}{2}$

$$(Q_\epsilon \Psi)(\underline{r}_1, \underline{r}_2) = \frac{\Psi(\underline{r}_1, \underline{r}_2) - \Psi(\underline{r}_2, \underline{r}_1)}{2}$$

Here any wave-function is sum of bosonic + fermionic part. Not true in general!

A case of frequent relevance is when we work with separable wave functions

$$\Psi(r_1, \dots, r_n) = \psi_1(r_1)\psi_2(r_2)\cdots\psi_n(r_n)$$

This specifically comes up when one works in the "Hartree approximation" (sometimes called "mean field approximation"

$$H = \sum_{i=1}^{n} \frac{P_i^2}{2m} + V(X_i) \qquad \text{(i.e., no interactions)}$$

Then energy eigenstates are products of single particle eigenstates.

Suppose particles should be bosonic or fermionic, though?

fermionic: $$Q_\epsilon \Psi(r_1, \dots, r_n) = \frac{1}{n!} \underbrace{\begin{vmatrix} \psi_1(r_1) & \psi_2(r_1) & \cdots & \psi_n(r_1) \\ \psi_1(r_2) & \psi_2(r_2) & \cdots & \psi_n(r_2) \\ \vdots & & & \\ \psi_1(r_n) & \psi_2(r_n) & \cdots & \psi_n(r_n) \end{vmatrix}}_{\text{"slater determinant"}}$$

bosonic: $$Q_\epsilon \Psi(r_1, \dots, r_n) = \frac{1}{n!} \operatorname{Perm} \begin{bmatrix} \psi_1(r_1) & \psi_2(r_1) & \cdots & \psi_n(r_1) \\ \psi_1(r_2) & \psi_2(r_2) & \cdots & \psi_n(r_2) \\ \vdots & & & \\ \psi_1(r_n) & \psi_2(r_n) & \cdots & \psi_n(r_n) \end{bmatrix}$$

permanent of a matrix is like determinant but no minus signs!

For $\psi_i(r)$ a basis for $L^2(\mathbb{R}^3)$, we have that $\psi_{i_1}(r_1)\cdots\psi_{i_n}(r_n)$ a basis for $L^2(\mathbb{R}^{3n})$.

So arbitrary bosonic/fermionic states can be written in terms of these separable state wave functions.
We see an immediate consequence of anti-symmetrization for fermions.

Thm: If any two wave functions ψ_i, ψ_j are the same, then $Q_\epsilon(\psi_1, \dots, \psi_i, \dots, \psi_j, \dots, \psi_n) = 0$

In words, we have Pauli exclusion principle: "Two fermions can never occupy the same quantum state"

Can consider bosonic / fermionic composite systems more generally:

n copies of system described by Hilbert space $\mathcal{H} \longrightarrow \mathcal{H}_{\text{composite}} = \mathcal{H}^{\otimes n}$

Have action $(S_n \times \otimes^n \mathcal{H}) \longrightarrow \otimes^n \mathcal{H}$

$$(\pi, e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_n}) \longrightarrow e_{\pi(i_1)} \otimes e_{\pi(i_2)} \otimes \dots \otimes e_{\pi(i_n)} \quad \{e_i \in \mathcal{H}\}$$

Can define same projectors: $Q_1(e_{i_1} \otimes \dots \otimes e_{i_n}) = \sum_{\pi \in S_n} \frac{1}{n!} e_{\pi(i_1)} \otimes \dots \otimes e_{\pi(i_n)}$, extended to $\otimes^n \mathcal{H}$ by linearity.

$Q_\epsilon(e_{i_1} \otimes \dots \otimes e_{i_n}) = \sum_{\pi \in S_n} \frac{1}{n!} \epsilon(\pi) e_{\pi(i_1)} \otimes \dots \otimes e_{\pi(i_n)}$, extended to $\otimes^n \mathcal{H}$ by linearity.

Q_1 projects onto "symmetric tensors", $\mathrm{Im}(Q_1) = \odot^n \mathcal{H}$ "n-boson Hilbert space"

Q_ϵ projects onto "antisymmetric tensors"; $\mathrm{Im}(Q_\epsilon) = \Lambda^n \mathcal{H}$ "n-fermion Hilbert space"

Suppose $\dim \mathcal{H} = N$, count states (separable basis)

$$\dim(\Lambda^n \mathcal{H}) = \begin{cases} \frac{N!}{(N-n)!\,n!} = \binom{N}{n} & \text{for } n \leq N \\ 0 & \text{for } n > N \end{cases}$$

Exercise (Thm 16.2.4 Hannabuss): $\dim(\odot^n \mathcal{H}) = \binom{N+n-1}{n}$