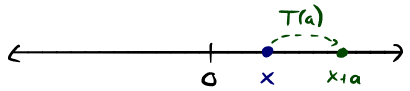
B7.3 Further quantum theory lecture 5

Now we turn to the subject of symmetries in quantum theory. We know from classical physics (dynamics, electromagnetism, etc.) that exact (or approximate) symmetries of a system can be an indispensible tool in their analysis.

(If you took classical mechanics, you know there is a deep theorem due to Noether that says that every continuous symmetry (translations, rotations, etc.) gives rise to a conserved quantity, sometimes called an "integral of motion".)

Before considering the subject formally, let's look at possibly the simplest example: translations in $\mathbb{R}$, which should be a symmetry of the free particle w/ Hamiltonian $H = P^2/2m$.

We define an operator $T(a)$ that translates the system by an amount a:



How does this act on wave-functions? It shifts argument by minus a:

$$\big(T(a)\psi\big)(x+a) = \psi(x) \Rightarrow \big(T(a)\psi\big)(x) = \psi(x-a)$$

$$T(a) : \psi(x) \longmapsto \tilde{\psi}(x) = \psi(x-a)$$

Can also see this using generalized position eigenstates: $T(a)|x\rangle = |x+a\rangle$, so

$$(T(a)\psi)(x) = \langle x|T(a)|\psi\rangle = \int dy\, \langle x|T(a)|y\rangle\langle y|\psi\rangle = \int dy\, \langle x|y+a\rangle \psi(y) = \int dy\, \delta(x-y-a)\psi(y) = \psi(x-a)$$

Observe some features: $T(a)T(b) = T(a+b)$

$T(a)^{-1} = T(-a)$

can work out adjoint operator

$$\left(\langle\phi|T(a)\psi\rangle = \int_{-\infty}^{\infty} dx\, \bar{\phi}(x)\psi(x-a) = \int_{-\infty}^{\infty} dx\, \bar{\phi}(x+a)\psi(x) = \langle T(-a)\phi|\psi\rangle \right)$$

$T(a)^* = T(-a) = T(a)^{-1}$ so $T(a)$ is a unitary operator on $\mathcal{H}$.

(in particular, $\langle T(a)\psi|T(a)\phi\rangle = \langle\psi|\phi\rangle$ so transition amplitudes are invariant.)

This interpretation of the translation operator is as an active transformation (move system to the right with respect to fixed reference frame). Can also see passive point of view, where we think of translating our reference frame by $-a$. In this case, it is the operators that transform:

$$\langle T(a)\phi|A|T(a)\psi\rangle = \langle\phi|T(a)^{-1}AT(a)|\psi\rangle$$

$$A \xmapsto{T(a)} T(a)^* A T(a)$$

$$X \xmapsto{T(a)} X + a$$

Will return to this re: time evolution (Schrödinger vs. Heisenberg picture).

Translations can be taken arbitrarily small, in which case the action of the translation operator should be arbitrarily close to that of the identity operator. Indeed, we have

$$\lim_{\epsilon\to 0} T(\epsilon)\Psi(x) = \lim_{\epsilon\to 0} \Psi(x-\epsilon) = \Psi(x) - \epsilon\Psi'(x) + O(\epsilon^2)$$

We interpret this as giving the infinitesimal expansion of the operator

$$T(\epsilon) = \mathbb{1} - \frac{i\epsilon}{\hbar} T_{inf} + O(\epsilon^2) \quad \text{where} \quad T_{inf}\Psi(x) = -i\hbar\Psi'(x)$$

In otherwords, $T_{inf} \equiv P$. We say P is the infinitesimal generator of translations because we can write a finite translation as a composite of many infinitesimals

$$T(a) = T\left(\tfrac{a}{N}\right)^N = \left(\mathbb{1} - \frac{iaP}{\hbar N} + O\left(\frac{a^2}{N^2}\right)\right)^N \xrightarrow[N\to\infty]{} \exp\left(\frac{-iaP}{\hbar}\right) \quad (\text{Exercise!})$$

[In Part 1A, you will have seen another version of this statement using Taylor expansions of wave functions.]

This is actually even easier to see in momentum space. Note that

$$T(a)|p\rangle = \int dx\, T(a)|x\rangle\langle x|p\rangle = \int dx\, |x+a\rangle e^{\frac{ipx}{\hbar}}$$

$$= \int dx\, |x\rangle e^{\frac{ip(x-a)}{\hbar}} = e^{\frac{-ipa}{\hbar}} \int dx\, |x\rangle e^{\frac{ipx}{\hbar}}$$

$$= \exp\left(\frac{-ipa}{\hbar}\right)|p\rangle = \exp\left(\frac{-iPa}{\hbar}\right)|p\rangle$$

So our operator identity holds on the momentum basis, thus for all states. Note P is self-adjoint, and this is compatible with unitarity of $T(a)$:

$$(T(a))^* = \left(\exp\left(\frac{-iPa}{\hbar}\right)\right)^* = \exp\left(\frac{iPa}{\hbar}\right) = T(a)^{-1}$$

From the passive perspective, we have $H \xmapsto{T(a)} T(a)^* H\, T(a) = \exp\left(\frac{iPa}{\hbar}\right) H \exp\left(\frac{-iPa}{\hbar}\right)$

$$= H \text{ for any } a \text{ iff } [P,H]=0.$$

So for, e.g., $H = \frac{P^2}{2m}$, $T(a)$ is a symmetry of the Hamiltonian, and this means we can find simultaneous eigenstates of H and P. Thus P is a conserved quantity under time evolution.

Abstracting a bit, what have we found out about the structure of symmetries, at least in the case of translation?

1 • Implemented via unitary (linear) operators on $\mathcal{H}$.

2 • Symmetries form a group (additive group on $\mathbb{R}$) and unitaries respect group law.

$$U(g_1)U(g_2) = U(g_1 g_2) \quad , \quad U(g^{-1}) = U(g)^{-1} = U(g)^*$$

[In other words, have unitary representation of $\mathbb{R}$ on $\mathcal{H}$; group homomorphism $\mathbb{R} \xrightarrow{T} U(\mathcal{H})$]

3 • Infinitesimal translation implemented by self-adjoint operator (momentum) which generates finite transformations via exponentiation.

$$U(g(\theta)) = \exp\left(\frac{-iG\theta}{\hbar}\right) , \quad G = G^*$$

4 • For symmetries of the Hamiltonian, inf. generator commutes with H and gives conserved quantity (momentum).

$$U(g^{-1})HU(g) = H \iff [G,H] = 0 \implies G \text{ conserved}$$

These properties come close to capturing the general situation, but there are subtleties. To start with, why aren't the first two points "obvious"? Unitaries are the natural isomorphisms of Hilbert spaces. Surely symmetries should act as such (and respect group structure. Right?

Recall that space of physical states is not $\mathcal{H}$, but $\mathbb{P}(\mathcal{H})$ (space of rays in $\mathcal{H}$). So really, a quantum symmetry need only be defined as a map

$$s : \mathbb{P}(\mathcal{H}) \longrightarrow \mathbb{P}(\mathcal{H})$$

that preserves transition probabilities:

$$\frac{|\langle\phi'|\psi'\rangle|^2}{\|\phi'\|^2\|\psi'\|^2} = \frac{|\langle\phi|\psi\rangle|}{\|\phi\|^2\|\psi\|^2} \qquad \text{where } s: \text{rays through } \phi,\psi \longmapsto \text{rays through } \phi',\psi'$$

takes same value for ϕ,ψ replaced by any vectors in same rays, so is a function on $\mathbb{P}(\mathcal{H}) \times \mathbb{P}(\mathcal{H})$.

Naively, seems this could be a much weaker condition on s than what is necessary for it to descend from a unitary map on $\mathcal{H}$. Situation explained completely by a theorem of Wigner:

Thm. (Wigner): Let G be a group that acts on $\mathbb{P}(\mathcal{H})$ by quantum symmetries. Then $\forall g \in G$, this action lifts to an operator $U(g): \mathcal{H} \to \mathcal{H}$ that is either unitary or anti-unitary. This lift is defined up to multiplication by a complex number of modulus 1.

Aside: an anti-unitary map $A: \mathcal{H} \to \mathcal{H}$ is a $\mathbb{C}$-antilinear map obeying $(A\phi, A\psi) = (\psi, \phi)$. An example is complex conjugation on $L^2(\mathbb{R})$.

Remark: if $A: \mathcal{H} \to \mathcal{H}$ is anti-unitary, then A^2 is unitary, so any symmetry that can be written as the square of another symmetry operation is necessarily represented as a unitary operator on $\mathcal{H}$.

Because the group action needs only be respected at the level of rays, we can have

$$U(g_1)U(g_2) = e^{i\phi(g_1,g_2)}U(g_1g_2) \quad , \quad \phi(g_1,g_2) \in [0,2\pi)$$

These must obey $e^{i\phi(g_1,g_2)}e^{i\phi(g_1g_2,g_3)} = e^{i\phi(g_1,g_2g_3)}e^{i\phi(g_2,g_3)}$ and they are defined up to rephasings. A realization of a group of this type is called a projective unitary representation. Another technical Lemma says that the phases $\phi(g_i,g_j)$ can be chosen to vanish for transformations in a neighborhood of the identity. It may not be possible globally, though, as we'll see in our discussions of spin.

Now we turn to the third point. This is actually completely general. There is a powerful theorem called Stone's thm. on one-parameter unitary groups that essentially guarantees that for any one-parameter family of unitaries

$$\{U_{t\in\mathbb{R}}\}, \; U_{t_1}U_{t_2} = U_{t_1+t_2}$$

there is a self-adjoint generator $G = G^*$ s.t. $U(t) = \exp\left(\frac{-itG}{\hbar}\right)$. Conversely, any self-adjoint G generates such a one-parameter family of unitaries.

> In physics texts, one often just asserts $U(t) = 1 - \frac{itG}{\hbar} + O(t^2)$, $U(t)^* = 1 + \frac{itG^*}{\hbar} + O(t^2)$ and confirms
>
> $$U(t)U^*(t) = 1 + \frac{it}{\hbar}(G^* - G) + O(t^2) \equiv 1 \implies G = G^*$$

Finally, to discuss dynamical symmetries more carefully, we reconsider the subject of time translation. We've introduced previously the time evolution operator $U(t_1,t_0)$. Now we can see that in time-translationally invariant theories, $U(t_1,t_0) \equiv U_{t_1-t_0}$ is the one-parameter family of unitaries guaranteed by Wigner's thm, and by Stone's thm. we have

$$U_t = \exp\left(\frac{-iHt}{\hbar}\right) \quad , \quad H = H^*$$

This could be taken as a definition of the Hamiltonian, H. Then we have

$$\Psi_t = U(t)\Psi_0$$
$$i\hbar\frac{d}{dt}\Psi_t = i\hbar\left(\frac{d}{dt}U(t)\right)\Psi_0 = H\Psi_0$$

thus recovering time-dependent Schrödinger equation.

A dynamical symmetry is should represent a transformation of the system such that the process of time evolution is invariant, i.e.,

$$(U\Psi)_t = e^{\frac{-iHt}{\hbar}}U\Psi = Ue^{\frac{-iHt}{\hbar}}\Psi = U(\Psi_t) \quad \forall \Psi \in \mathcal{H}$$
$$\iff U^{-1}e^{\frac{-iHt}{\hbar}}U = e^{\frac{-iHt}{\hbar}} \iff U^{-1}HU = H$$

if $U = \exp\left(\frac{-iG\theta}{\hbar}\right)$ then equivalently, $[G,H] = 0$, so generator is a conserved quantity.

So to sum up what we have for general quantum symmetries:

1. Implemented via unitary or anti-unitary operators on $\mathcal{H}$.

2. Unitarily symmetries realized by projective unitary representation (group law obeyed "up to a phase").

3. Infinitesimal translation implemented by self-adjoint operator (momentum) which generates finite transformations via exponentiation.

4. For dynamical symmetries we have (equivalently)

- $[G,H]=0$ (H & G simultaneously diagonalizable)
- $G_t = U_t^* G U_t = G$ (G invariant under time evolution)
- $\exp\left(\frac{iG\theta}{\hbar}\right) H \exp\left(\frac{-iG\theta}{\hbar}\right) = H$ (Hamiltonian invariant under symmetry group action)
- $\exp\left(\frac{-iG\theta}{\hbar}\right) U_t = U_t \exp\left(\frac{-iG\theta}{\hbar}\right)$ (Symmetry and time evolution commute)

So far, we considered only Abelian symmetries as examples. Next we turn to rotations, which are realized by the non-Abelian group $SO(3)$.