B7.3 Further Quantum Theory Lecture 7

Last time, we were reviewing the action of rotations on quantum systems and consequently the representation theory of the angular momentum operators.

$$SO(3) \circlearrowright \mathcal{H}$$
$$\underset{\Psi}{R_{\hat{n}}(\theta)} \longmapsto U(R_{\hat{n}}(\theta)) = \exp\left(\frac{-i\theta}{\hbar}\underline{J}\cdot\hat{\underline{n}}\right)$$
$$[J_i, J_j] = i\hbar\sum_k \epsilon_{ijk}J_k \longrightarrow [\underline{\omega}\cdot\underline{J}, \underline{\omega}'\cdot\underline{J}] = i\hbar(\underline{\omega}\times\underline{\omega}')\cdot\underline{J}$$

We reviewed the classification of irreducible representations of the angular momentum operators

$$\forall\, j \in \tfrac{1}{2}\mathbb{Z}_{\geq 0} \quad \mathcal{H}_{\text{spin-}j} \cong \mathbb{C}^{2j+1} \text{ with orthonormal basis } |j,m\rangle,\ m\in\{-j,-j+1,\dots,j-1,j\}$$
$$J_\pm|j,m\rangle = \hbar\sqrt{j(j+1)-m(m\pm1)}\,|j,m\pm1\rangle$$

A particularly interesting representation, for various reasons that we shall see, is the spin-$\frac{1}{2}$ representation, $\mathcal{H}_{\text{spin-}\frac{1}{2}} \cong \mathbb{C}^2$. This is the smallest nontrivial representation, and can be given explicitly in terms of the Pauli spin matrices:

$$\sigma_1 = \begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2 = \begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \sigma_3 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

Defining $J_i = \frac{\hbar}{2}\sigma_i$, one can verify directly that the J_i obey the A.M. commutation relations. Note that the ladder operators now take the simple for

$$J_+ = \hbar\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad J_- = \hbar\begin{pmatrix}0&0\\1&0\end{pmatrix}$$

There are many conventions for the naming of basis vectors in this case: $\begin{pmatrix}1\\0\end{pmatrix} \equiv |\tfrac{1}{2},\tfrac{1}{2}\rangle \equiv |+\rangle \equiv |\uparrow\rangle$

$$\begin{pmatrix}0\\1\end{pmatrix} \equiv |\tfrac{1}{2},-\tfrac{1}{2}\rangle \equiv |-\rangle \equiv |\downarrow\rangle$$

So here the normalizations are easy, $J_+|-\rangle = \hbar|+\rangle$, $J_-|+\rangle = \hbar|-\rangle$.

Before discussing the $SO(3)$ action on this representation, a comment may be in order. Along with the identity matrix, the Pauli matrices form a basis for Hermitian (self-adjoint) 2×2 matrices, so the language of angular momentum & spin is ubiquitous whenever one is working with 2-state quantum systems (qubits).

Now for any unit vector $\hat{n}$ and angle $\theta\in[0,2\pi)$ we can write the 2×2 unitary matrix that implements the rotation $R_{\hat{n}}(\theta)$ on $\mathcal{H}_{\text{spin-}\frac{1}{2}}$:

$$U(R_{\hat{n}}(\theta)) = \exp\left(\frac{-i}{\hbar}(\hat{\underline{n}}\cdot\underline{J})\theta\right) = \exp\left(\frac{-i\underline{\sigma}\cdot\hat{\underline{n}}\,\theta}{2}\right)$$

On the problem sheet you will show that this can be written as

$$U(R_{\hat{n}}(\theta)) = \cos\left(\tfrac{\theta}{2}\right)\mathbb{1} - i\sin\left(\tfrac{\theta}{2}\right)\underline{\sigma}\cdot\hat{\underline{n}}$$

This is the most general matrix in $SU(2)\equiv\{2\times2$ unitary matrices with determinant $=1\}$. However, there is a subtlety: as $\theta\to2\pi$, the matrix approaches $-\mathbb{1}_{2\times2}$ rather than the identity! What's going on? Look at it this way.

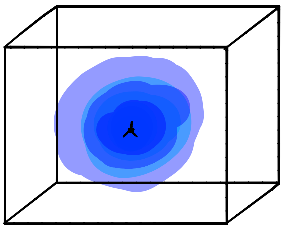
$$\begin{aligned} U(R_{\hat{n}}(\pi))\,U(R_{\hat{n}}(\pi)) &= -U(R_{\hat{n}}(2\pi)) \\ &= -U(R_{\hat{n}}(0)) \end{aligned}$$

So what we have is one of our projective unitary representations. Physicists say "under rotation by 2π, state comes back to itself up to a phase" (in this case a sign).

As $\theta \in [0, 4\pi)$, our $\mathcal{U}(R_{\hat{n}}(\theta))$ goes through all possible $SU(2)$ matrices while double-counting $SO(3)$ rotations. We say that $SU(2)$ is a double-cover of $SO(3)$ (in fact the universal cover). A theorem of Bargmann (1954) shows that proj. unitary rep's of G are equivalent to unitary reps of $\tilde{G}$ (universal cover of G). So quantum realizations of rotational symmetry must arise through unitary reps of $SU(2)$.

The unitary $SU(2)$ reps that give honest $SO(3)$ reps are exactly those w/ integer spin; half-integer spins all have analogous phase as in spin-1/2. This is why only integer spins appear in $L^2(S^2)$. There we manifestly have an $SO(3)$ action on wavefunctions (inherited from the action on coordinates).

Now we'll move on to the problem of combining angular momentum. This is particularly relevant due to the phenomenon of spin for elementary (and composite) particles, where intrinsic degrees of freedom transform under rotations.



$$\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathcal{H}^{(\text{internal})}$$

$\circlearrowleft \quad \circlearrowleft$

$SO(3)$ Rotations

Here $\Psi(\underline{x}) \otimes \phi \xrightarrow{R} \Psi(R\underline{x}) \otimes U(R)\phi$. We assume $\mathcal{H}^{(\text{int})} \cong \mathcal{H}_{\text{spin-}j}$ for some spin $j \in \frac{1}{2}\mathbb{Z}_{\geq 0}$.
On the full Hilbert space, we have two commuting copies of the angular momentum operators.

- $\underline{L} = \underline{X} \wedge \underline{P}$ act as identity on $\mathcal{H}^{(\text{int})}_{\text{spin-}j}$.
- $\underline{S}$ act on $\mathcal{H}^{(\text{int})}_{\text{spin-}j}$ nontrivially, but as identity on $L^2(\mathbb{R}^3)$.

In general, wouldn't expect $\underline{L}$ & $\underline{S}$ to individually commute w/ Hamiltonian, but in rotationally symmetric setting the sum should:

$$\Psi(x) \otimes \phi \longmapsto \left(\Psi(x) - \tfrac{i}{\hbar}\epsilon(\underline{L}\cdot\hat{n})\Psi(x)\right) \otimes \left(\mathbb{1} - \tfrac{i}{\hbar}\epsilon(\underline{S}\cdot\hat{n})\phi\right) \approx \Psi(\underline{x}) \otimes \phi - \tfrac{i\epsilon}{\hbar}\left((\underline{L} + \underline{S})\cdot\hat{n}\right)\Psi(\underline{x}) \otimes \phi$$

So a natural and important question is how we can understand the action of the "total" angular momentum operators on the tensor product Hilbert space.

$$J_i^{(\text{tot})} = J_i^{(1)} + J_i^{(2)} = \underbrace{L_i}_{\text{orbital ang. momentum}} + \underbrace{S_i}_{\text{spin angular momentum}}$$

If we have already understood the decomposition of our original Hilbert space into irreps of $\underline{L}$ and $\underline{S}$ (say, in terms of spherical harmonics for $\underline{L}$), this reduces to the problem of combining two irreps.

$$\mathcal{H} \cong \underset{\substack{\circlearrowleft \\ J_i^{(1)}}}{\mathcal{H}_{\text{spin-}j_1}} \otimes \underset{\substack{\circlearrowleft \\ J_i^{(2)}}}{\mathcal{H}_{\text{spin-}j_2}}$$

First let's look at a simple case: $j_1 = j_2 = \frac{1}{2}$. Then we have a basis for the tensor product

$$\begin{array}{r|cc}
m_2 = \frac{1}{2} & |{+}{+}\rangle \xrightarrow{J_-^{(1)}} & |{-}{+}\rangle \\
 & \downarrow J_-^{(2)} & \downarrow J_-^{(2)} \\
m_2 = -\frac{1}{2} & |{+}{-}\rangle \xrightarrow{J_-^{(1)}} & |{-}{-}\rangle \\
\hline
 & m_1 = \frac{1}{2} & m_1 = -\frac{1}{2}
\end{array}$$

These states are eigenstates for: $(J^{(1)})^2$, $(J^{(2)})^2$, $(J_3^{(1)})$, $(J_3^{(2)})$. What about the action of $J_i^{(tot)}$? Note that $(J^{(tot)})^2 = (J^{(1)})^2 + (J^{(2)})^2 + 2\underline{J}^{(1)} \cdot \underline{J}^{(2)}$ clearly commutes with $(J^{(1)})^2$ and $(J^{(2)})^2$, but not $J_3^{(1)}$ or $J_3^{(2)}$. If we let $J_3^{(tot)} = \hbar M$ on eigenstates, and $(J^{(tot)})^2 = \hbar^2 J(J+1)$ on eigenstates, then we have the following rearrangement:

$$\begin{array}{c|ccc}
 & M = 1 & M = 0 & M = -1 \\
\hline
J = 1 & 0 \xleftarrow{J_+^{(tot)}} |{+}{+}\rangle & \xrightarrow{J_-^{(tot)}} |{-}{+}\rangle + |{+}{-}\rangle & \xrightarrow{J_-^{(tot)}} |{-}{-}\rangle \xrightarrow{J_-^{(tot)}} 0 \\
J = 0 & & 0 \xleftarrow{J_+^{(tot)}} |{-}{+}\rangle - |{+}{-}\rangle \xrightarrow{J_-^{(tot)}} 0 & \\
\hline
\end{array}$$

$$\mathcal{H}_{\text{spin-}1/2} \otimes \mathcal{H}_{\text{spin-}1/2} \cong \mathcal{H}_{\text{spin-}1} \oplus \mathcal{H}_{\text{spin-}0}$$

$$\left.\begin{array}{l} \odot^2 \mathcal{H}_{\text{spin-}1/2} \cong \mathcal{H}_{\text{spin-}1} \\ \Lambda^2 \mathcal{H}_{\text{spin-}1/2} \cong \mathcal{H}_{\text{spin-}0} \end{array}\right\} \text{from (anti-)symmetry above.}$$

The idea is that on $\mathcal{H}_1 \otimes \mathcal{H}_2$, we have two inequivalent choices of (quadruples of) observables to diagonalize:

$$\{(J^{(1)})^2, (J^{(2)})^2, J_3^{(1)}, J_3^{(2)}\} \longrightarrow \text{basis states } |j_1, m_1; j_2, m_2\rangle$$

Alternatively (and oftentimes more naturally) we can choose to trade $J_3^{(1)}$ for $J_3^{(tot)}$ & $(J^{(tot)})^2$:

$$\{(J^{(tot)})^2, J_3^{(tot)}, (J^{(1)})^2, (J^{(2)})^2\} \longrightarrow |j_1, j_2; J, M\rangle \equiv |J, M\rangle$$

(In the above example, j_1 & j_2 come along for the ride because we chose our original reps to be irreducible. More generally if we take $\mathcal{H}_1 \otimes \mathcal{H}_2$ with either reducible, j_1 & j_2 become meaningful labels.)

Coming back to the general case, first question is how tensor product decomposes into $\underline{J}^{(tot)}$ irreps (i.e., what values of J are allowed).

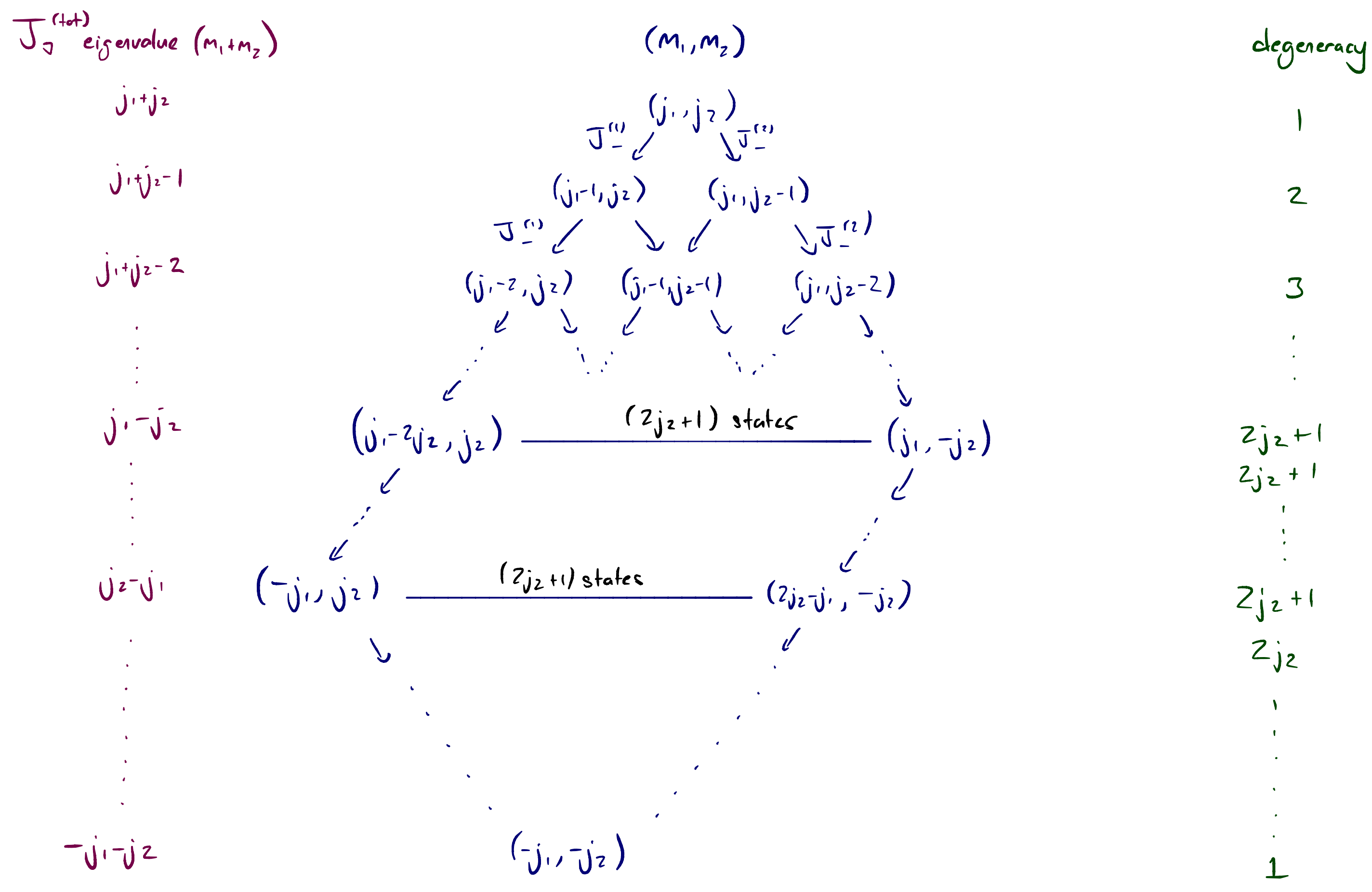
Observe that in $\mathcal{H}_1 \otimes \mathcal{H}_2$, the state $|j_1, j_1 ; j_2, j_2\rangle$ is the unique state with maximal $J_3^{(tot)}$ eigenvalue (it is $(j_1+j_2)\hbar$). Thus we can find a spin-(j_1+j_2) subrepresentation

$$
\begin{array}{c}
|j_1, j_1 ; j_2, j_2\rangle \\
\downarrow J_-^{(tot)} \\
\vdots \\
\downarrow J_-^{(tot)} \\
|j_1, -j_1 ; j_2, -j_2\rangle
\end{array}
$$

Looking at remaining states, maximum $J_3^{(tot)}$ eigenvalue will be j_1+j_2-1, so iterate. Can find the general pattern without too much difficulty:

Prop: $\mathcal{H}_{\text{spin-}j_1} \otimes \mathcal{H}_{\text{spin-}j_2} \cong \bigoplus_{J=|j_1-j_2|}^{j_1+j_2} \mathcal{H}_{\text{spin-}J}$

Proof: diagrammatic (wlog $j_2 \leq j_1$) WLOG $j'' \leq j'$



Working from the top/bottom: $\mathcal{H} = \mathcal{H}_{j_1+j_2} \oplus \mathcal{H}_{j_1+j_2-1} \oplus \cdots \oplus \mathcal{H}_{j_1-j_2} = \bigoplus_{J=|j_1-j_2|}^{j_1+j} \mathcal{H}_{\text{spin-}J}$

For fun can check dimensionalities agree.