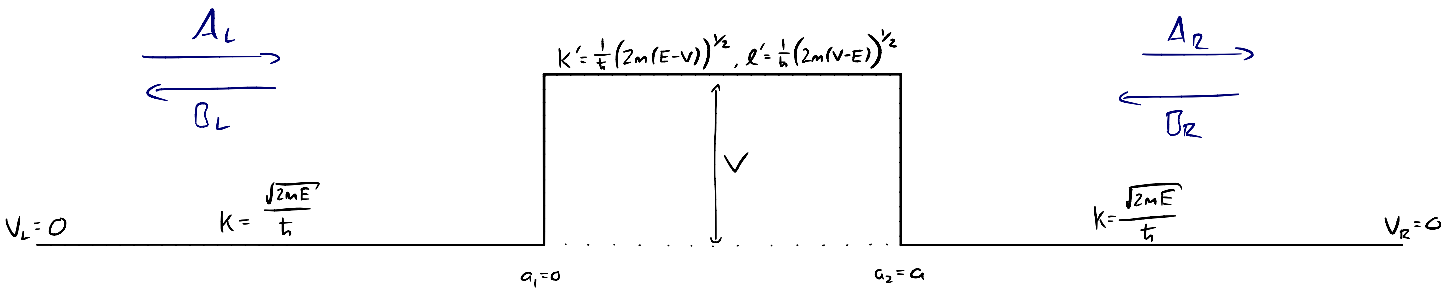


$$s = s_1 = s_2 = K + K'$$
$$d = d_1 = -d_2 = K - K'$$

$$\begin{pmatrix} A_L \\ B_L \end{pmatrix} = M \begin{pmatrix} A_R \\ B_R \end{pmatrix} \quad \text{where} \quad M = M_1 M_2 = \frac{1}{4KK'} \begin{pmatrix} s^2 e^{ida} - d^2 e^{isa} & sd(e^{-ida} - e^{-isa}) \\ sd(e^{ida} - e^{isa}) & s^2 e^{-ida} - d^2 e^{-isa} \end{pmatrix}$$

$$M_{11} = \frac{s^2 e^{ida} - d^2 e^{isa}}{(s^2 - d^2)} = e^{ika}\left[\frac{-(k^2+k'^2)(2i\sin(k'a)) + 2kk'(\cos(k'a))}{4kk'}\right] \longrightarrow |M_{11}|^2 = \frac{(k^2+k'^2)^2 \sin^2(k'a) + k^2k'^2(4\cos^2(k'a))}{4k^2k'^2}$$

$$M_{21} = \frac{sd(e^{ida} - e^{isa})}{(s^2 - d^2)} = e^{ika}\left[\frac{(k^2-k'^2)(-2i\sin(k'a))}{4kk'}\right] \longrightarrow |M_{21}|^2 = \frac{\sin^2(k'a)(k^2-k'^2)^2}{4k^2k'^2}$$

When $B_R \equiv 0$, we have:

$$T = \frac{|A_R|^2}{|A_L|^2} = \frac{1}{|M_{11}|^2} = \frac{4k^2k'^2}{(k^2+k'^2)^2\sin^2(k'a) + 4k^2k'^2\cos^2(k'a)} \quad \& \quad R = \frac{|B_L|^2}{|A_R|^2} = \frac{|M_{21}|^2}{|M_{11}|^2} = \frac{(k^2-k'^2)^2\sin^2(k'a)}{(k^2+k'^2)^2\sin^2(k'a) + 4k^2k'^2\cos^2(k'a)}$$

K and k' are implicitly functions of E, so this gives amplitudes for reflection and transmission as functions of E.
When we take $E < V_0$, $k' \to i\ell'$ and we get the modification (have to track factors of i)

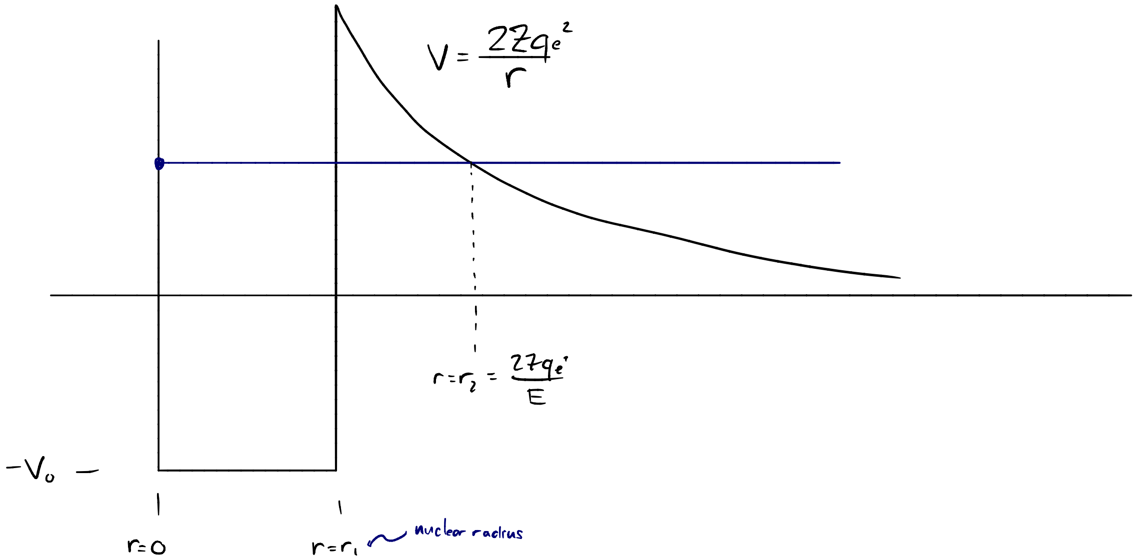
$$T = \frac{1}{|M_{11}|^2} = \frac{4k^2\ell^2}{(k^2-\ell^2)^2\sinh^2(\ell a) + 4k^2\ell^2\cosh^2(\ell a)} \quad \& \quad R = \frac{|M_{21}|^2}{|M_{11}|^2} = \frac{(k^2+\ell^2)^2\sinh^2(\ell a)}{(k^2-\ell^2)^2\sinh^2(\ell a) + 4k^2\ell^2\cosh^2(\ell a)}$$

Non-zero transmission coefficient T in this range describes "quantum tunnelling". For $\ell a \gg 1$, we have

$$T \approx \frac{16k^2\ell^2}{(k^2+\ell^2)^2} e^{-2\ell a} \sim e^{-2\ell a} \qquad R \approx 1 + O(e^{-2\ell a})$$

The probability of penetrating the barrier falls off exponentially over a characteristic length scale $\frac{1}{\ell} = \frac{\hbar}{\sqrt{2m(V-E)}}$. The $\hbar$ shows that for macroscopic E & V, this will be a tiny distance.

This order-of-magnitude estimate combines well with the WKB approximation. A famous example is Gamow's model of nuclear decay.



An α-particle (2 protons + 2 neutrons) is modelled as a particle in a square well that, for $r > r_1$, is replaced with a repulsive Coulomb potential. We estimate the tunnelling probability using WKB and the approximation that the exponential decay factor dominates the process.

$$|T|^2 \sim e^{-2\gamma}, \quad \gamma = \frac{1}{\hbar}\int_{r_1}^{r_2} \sqrt{2m(V-E)} = \frac{\sqrt{2mE}}{\hbar}\int_{r_1}^{r_2}\left(\frac{r_2}{r}-1\right)^{1/2} = \frac{\sqrt{2mE}}{\hbar}\left[r_2 \cos^{-1}\left(\sqrt{\frac{r_1}{r_2}}\right) - \sqrt{r_1(r_2-r_1)}\right]$$

$$\underset{r_2 \gg r_1}{\approx} \frac{\sqrt{2mE}}{\hbar}\left[\frac{\pi}{2}r_2 - 2\sqrt{r_1 r_2}\right] = \alpha\frac{Z}{\sqrt{E}} - \beta\sqrt{Zr_1}$$

$$\left\{\begin{array}{l} \alpha = \frac{q_e^2 \pi \sqrt{2m}}{\hbar} \approx 1.98 \text{ MeV}^{1/2} \\ \beta = \frac{4\sqrt{m}}{\hbar} \approx 1.485 \text{ fm}^{-1/2} \end{array}\right\}$$

Now we imagine (crudely) that when in the nucleus, the α-particle bouncing around and colliding with the potential wall with some frequency (say $v/2r_1$ if v is the "velocity" in the nucleus). Then the tunnelling rate (a.k.a. decay constant)

$v = \sqrt{2(E+V_0)/m}$

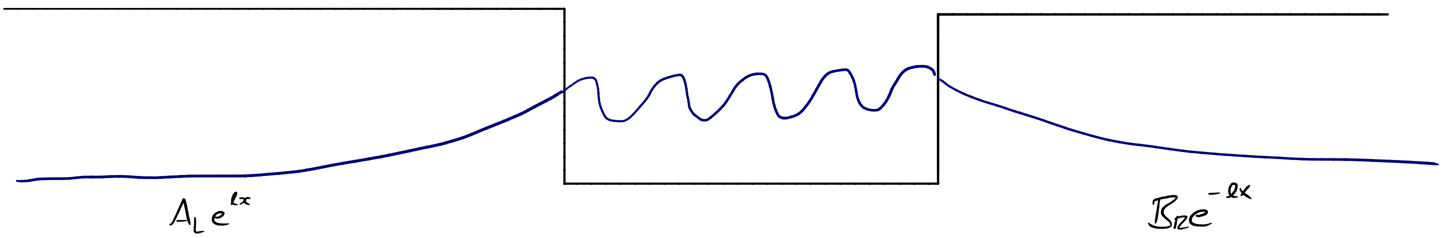
$$\lambda \sim \frac{v}{r_1} e^{-2\gamma}$$

in which case the half-life of the nucleus (the time after 50% of nuclei should decay) takes the form

$$t_{1/2} = \frac{\log(2)}{\lambda} \sim \frac{r_1 \log(2)}{\sqrt{\frac{2}{m}(E+V_0)}} e^{\frac{2\alpha Z}{\sqrt{E}} - 2\beta\sqrt{Zr_1}}$$

this result is massively dominated by the exponential dependence on $\frac{1}{\sqrt{E}}$. In experiments, by looking at different unstable isotopes of, say, Uranium (with $Z=92$), one has $E = 4.2$ MeV $-$ 6.7 MeV. This leads to values of $t_{1/2}$ that vary between 12 minutes and $4.5 \cdot 10^9$ years, 16 orders of magnitude!!!

If we continue our results to negative E (in particular, $V_0 < E < 0$), we should be able to say something about bound states. Now we have $k \to -i\ell$ and $k' \in \mathbb{R}$.



Now if we look at the defining relation for the S-matrix, we see

$$S \begin{pmatrix} A_L \\ B_R \end{pmatrix} = \begin{pmatrix} A_R \\ B_L \end{pmatrix} \overset{!}{=} 0 \quad \text{for normalizable bound state.}$$

So zeroes of S-matrix at $k = -i\ell \in -i\mathbb{R}_{\geq 0}$ encode bound states of the potential! For us, this requires $\det(S) = 0$, where

$$S = \begin{pmatrix} \frac{1}{M_{11}} & \frac{-M_{12}}{M_{11}} \\ \frac{M_{21}}{M_{11}} & \frac{\det M}{M_{11}} \end{pmatrix} \implies \det(S) = \frac{M_{22}}{M_{11}}$$

We can compute $\displaystyle\frac{M_{22}}{M_{11}} = \frac{s^2 e^{-ida} - d^2 e^{-isa}}{s^2 e^{ida} - d^2 e^{isa}} \xrightarrow{k=-i\ell} -e^{-2a\ell} \left\{ \frac{(k'^2-\ell^2)\sin(k'a) - 2k'\ell\cos(k'a)}{(k'^2-\ell^2)\sin(k'a) + 2k'\ell\cos(k'a)} \right\} = 0$

So bound states encoded in sol'ns to $\displaystyle\frac{2k'\ell}{(k'^2-\ell^2)} = \tan(k'a)$ where $\ell^2 + k'^2 = -2mV_0$. Similarly if we send $k \to +i\ell$ with $\ell \in \mathbb{R}$, then the non-zero wave-function in the forbidden regions have B_L & A_R as non-zero coefficients. So (S^{-1}) will have to have zero eigenvalues. These are encoded in poles in S. Indeed, for $k = i\ell$,

$$\frac{M_{22}}{M_{11}} \xrightarrow{k=i\ell} -e^{2a\ell} \left\{ \frac{(k'^2-\ell^2)\sin(k'a) + 2k'\ell\cos(k'a)}{(k'^2-\ell^2)\sin(k'a) - 2k'\ell\cos(k'a)} \right\}$$

and poles are in same positions as zeroes previously.

This is a shadow of a very general phenomenon in quantum mechanical scattering theory, where analytic properties of scattering amplitudes encode a wealth of physical information. This is even more the case in relativistic scattering theory/QFT.

A bit on further directions: quantum theory now has an extraordinarily diverse set of applications and connections to other subjects in pure and applied mathematics and physics. I want to mention a few:

- Geometric quantization: start with a general phase space (symplectic manifold); construct Hilbert space and algebra of observables. Important connections to topics in "geometric representation theory" and differential geometry.
- Semi-classical analysis: whole branch of PDE coming out of WKB methods. Interesting interplay with complex analysis (Stokes' phenomena, monodromies, etc.)
- Rigorous perturbation theory: a branch of functional analysis. More generally, a more detailed functional-analytic treatment of quantum theory in terms of spectral theory puts much of what we've done on much firmer footing.
- Quantum computing: what can you do with qubits? Manipulating superpositions is more powerful than manipulating just pure tensor states (i.e. classical bits). Very of-the-moment!
- Relativistic QM/QFT: What happens when a particle can be changed into other particles (or photons)? How to incorporate $E=mc^2$? It's a major change of perspective. All particles are fluctuations of space-filling quantum fields (generalizing photons/EM field).

And much more. I hope you'll continue to investigate the subject!