**Electromagnetism, Sheet 1: Introduction to Electrostatics**

**Practice with Vectors**

You need to revise vectors from Mods. Here are a few relevant revision questions which you need to be able to do, but which we won't go through in the class:

(i) Prove the following:

$$\begin{aligned}
\vec{a} \wedge (\vec{b} \wedge \vec{c}) &= (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} \\
\nabla \wedge (\psi \vec{V}) &= \psi \nabla \wedge \vec{V} + \nabla \psi \wedge \vec{V} \\
\nabla \cdot (\vec{B} \wedge \vec{C}) &= \vec{C} \cdot \nabla \wedge \vec{B} - \vec{B} \cdot \nabla \wedge \vec{C} \\
\nabla \wedge (\vec{A} \wedge \vec{B}) &= \vec{A}(\nabla \cdot \vec{B}) + (\vec{B} \cdot \nabla)\vec{A} - \vec{B}(\nabla \cdot \vec{A}) - (\vec{A} \cdot \nabla)\vec{B}
\end{aligned}$$

Show that also

$$\nabla \wedge (\nabla \wedge \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

where $\nabla^2 \vec{A} = (\nabla^2 A_1, \nabla^2 A_2, \nabla^2 A_3)$ in cartesian coordinates.

(ii) Show that

$$\nabla \wedge \nabla \phi = 0$$

for any differentiable function $\phi$, and that conversely, if $\vec{E}$ is a differentiable vector field with $\nabla \wedge \vec{E} = 0$ in a simply connected region of space then there exists a function $\phi$ for which $\vec{E} = \nabla \phi$. What changes in $\phi$ leave $\vec{E}$ unchanged? (i.e. if $\vec{E} = \nabla \phi_1 = \nabla \phi_2$ what can you say about $\phi_1 - \phi_2$?)

(iii) Show that

$$\nabla \cdot \nabla \wedge \vec{A} = 0$$

for any differentiable vector field $\vec{A}$. One can also prove the converse: if $\vec{B}$ is a differentiable vector field with $\nabla \cdot \vec{B} = 0$ in a suitable region of space then there exists a vector field $\vec{A}$ for which $\vec{B} = \nabla \wedge \vec{A}$. Assuming for now that this is true, what changes in $\vec{A}$ leaves $\vec{B}$ unchanged? (You need (ii) above for this.)

(iv) If $\vec{E}$ is a differentiable vector field satisfying

$$\int \vec{E} \cdot d\mathbf{S} = 0$$

for every closed surface $S$ in a region $R$ of space, use the divergence theorem to show that $\nabla \cdot \vec{E} = 0$ in $R$.

If $\vec{B}$ is a differentiable vector field satisfying

$$\oint_\Gamma \vec{B} \cdot d\ell = 0$$

for every closed curve $\Gamma$ in a region $R$ of space, use Stoke's theorem to show that $\nabla \wedge \vec{B} = 0$ in $R$.

**Now some electromagnetism: classwork starts here**

1.- In 3-dimensional Cartesian coordinates define $\vec{r} = (x, y, z)$ and $r = |\vec{r}|$, as usual. Show that $\nabla \cdot \vec{r} = 3$ and that, if $r \neq 0$, $\nabla r = \frac{\vec{r}}{r}$. Deduce that, where $r \neq 0$,

$$\nabla^2 f(r) = f'' + \frac{2}{r} f' = \frac{1}{r}(rf)''$$

(so that's two things to prove; prime is $d/dr$)

Define $\vec{E} = -\nabla\left(\frac{k}{r}\right)$ for constant $k$. Show that, where $r \neq 0$, $\nabla \wedge \vec{E} = 0$ (clearly) and $\nabla \cdot \vec{E} = 0$, while

$$\int_S \vec{E} \cdot d\mathbf{S} = 4\pi k$$

where the surface integral is taken over any closed surface $S$ which includes the origin. *(First take $S$ to be a sphere with the origin as center, then use the Divergence Theorem to obtain the results for more general $S$.)*

2.- Consider two opposite charges, of magnitudes $q$ and $-q$, separated by the vector $\vec{d}$. Compute the scalar potential at all points in space, in the limit in which $q$ becomes very large and $d$ very small, with $\vec{p} \equiv q\vec{d}$ kept constant. Such configuration is called a dipole.

From the dipole potential compute the electric field at all points in space with $r \neq 0$. What do you expect the net charge of the configuration to be? check this result through the Gauss law for a sphere centered on the dipole.

3.- Consider an infinite cylinder of radius $a$ and uniform charge density $\rho$. Find the scalar potential at all points in space.

4.- You are told that in cylindrical polar coordinates $(R, \theta, z)$ the charge density is

$$\rho(R, \theta, z) = \frac{q}{2\pi R}\delta(R - a)\delta(z) \tag{1}$$

What does this charge density correspond to? compute the potential for all points in space with $R = 0$. Can you compare this with anything you have seen before?

5.- In spherical coordinates the time-average potential of a neutral hydrogen atom is given by

$$\Phi(\vec{r}) = \frac{q}{4\pi\epsilon_0}\frac{e^{-\alpha r}}{r}\left(1 + \frac{\alpha r}{2}\right) \tag{2}$$

Find the distribution of charge that will result in this potential and interpret your result physically.