Electromagnetism, Sheet 3: Magnetostatics

1.- (i) A point particle P of charge Ze is fixed at rest at the origin in 3-dimensions, while a point particle E of mass m and charge $-e$ moves in the electric field of P. Write down Newton's equation of motion for E, and show that the orbit lies in a plane. If the orbit is circular, with P at the centre, find the orbital frequency ω in terms of its radius a (together with Z, e, m and ϵ_0).

(ii) A point particle P of mass m and charge e moves with velocity $\vec{v}$ in a constant magnetic field $\vec{B} = B\vec{k}$, where $\vec{k}$ is the unit vector parallel to the $z-$axis and B is a constant (not necessarily positive). Write down Newton's equations of motion for P assuming the force on the particle is $e\vec{v} \wedge \vec{B}$, and show that $\vec{v} \cdot \vec{k}$ and $v^2 = \vec{v} \cdot \vec{v}$ are constants of motion.

Show that, if $\vec{v} \cdot \vec{k} = 0$, then P follows a circular path in a plane of constant z, with speed v and radius a related by $v = \frac{ea|B|}{m}$ (so that the speed can be arbitrarily large).

What is the frequency with which the circles are described? What is the path if $\vec{v} \cdot \vec{k} \neq 0$?

2.- In 3-dimensional Cartesian coordinates, define $R = (x^2 + y^2)^{1/2}$ and $\vec{A} = (0, 0, -k \log R)$. Show that

$$\nabla \cdot \vec{A} = 0, \qquad \nabla^2 \vec{A} = 0$$

Now define $\vec{B} = \nabla \wedge A$. Write out $\vec{B}$ explicitly in coordinates. Deduce (preferably not using this coordinate expression) that, where $R \neq 0$,

$$\nabla \cdot \vec{B} = 0, \qquad \nabla \wedge \vec{B} = 0$$

Use the coordinate expression for $\vec{B}$ to show that

$$\oint_\Gamma \vec{B} \cdot d\ell = 2\pi k$$

where the line-integral is taken along any close curve Γ which winds once around the $z-$axis (anticlockwise). *(First take Γ to be a circle in $z = 0$ centre at the origin, then uses Stoke's theorem to obtain the result for general Γ)*

3.- Assume $r \neq 0$ throughout this question. Define the vector field $\vec{A}$ by

$$\vec{A} = \frac{1}{r^3} \vec{k} \wedge \vec{r}$$

where $\vec{k}$ is the unit vector along the $z-$axis and $r, \vec{r}$ are the usual things. Show that $\vec{A}$ can also be written as $\nabla\phi \wedge \vec{k}$ with $\phi = 1/r$. Use the results from the revision section above to show that $\nabla \cdot \vec{A} = 0$ and that if $\vec{B} = \nabla \wedge \vec{A}$ then $\vec{B}$ can be written as

$$\vec{B} = \nabla \wedge \vec{A} = (\vec{k} \cdot \nabla)\nabla \frac{1}{r} = \nabla \left(\vec{k} \cdot \nabla \frac{1}{r} \right).$$

Deduce that $\nabla \cdot \vec{B} = 0$ (clearly) and that $\nabla \wedge \vec{B} = 0$. Show that $|\vec{B}| = \mathcal{O}(r^{-3})$ for large r, and deduce that the integral

$$\int_S \vec{B} \cdot d\mathbf{S} = 0,$$

where S is a sphere centered at the origin and having radius r, tends to zero as $r \to \infty$. Now explain why this integral over any closed S is zero (so that this field has zero magnetic charge, even at the origin where it is singular.)

4.- Suppose that $\vec{B}$ is a differentiable vector field defined everywhere and with zero divergence: $\nabla \cdot \vec{B} = 0$. Define a vector field $\vec{A}$ by the following integral

$$A_1(x, y, z) = \int_0^1 \lambda \left(z B_2(\lambda x, \lambda y, \lambda z) - y B_3(\lambda x, \lambda y, \lambda z) \right) d\lambda$$

together with its two cyclic permutations for A_2 and A_3. (Here $\vec{A} = (A_1, A_2, A_3)$ etc. This integral gives $\vec{A}$ at a point P as an integral along the straight line joining P to the origin.)

Show that

$$\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} = B_3$$

And deduce that $\nabla \wedge \vec{A} = \vec{B}$.

Proposed trick: Calculate the left-hand and do the integral; you need to find another formula for $\frac{d}{d\lambda} B_i(\lambda x, \lambda y, \lambda z)$ *and see how to use it.*

5.- A circular wire Γ of radius a lies in the plane $z = 0$ in Cartesian coordinates. Current I flows anticlockwise around Γ. Use the following equations from the lectures:

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \oint \frac{d\ell \wedge (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \qquad (*)$$

to calculate $\vec{B}$ at a point P on the $z-$axis, distance b from the origin. *(Easier than it looks if you first use symmetry to argue that* $\vec{B}$ *can only be in the* $\vec{k}$ *direction, then dot-product* $(*)$ *with* $\vec{k}$ *before calculating.)*

Now use $(*)$ above to obtain

$$\vec{B}(x, y, z) = \frac{\mu_0 I}{2\pi} \left(-\frac{y}{R^2}, \frac{x}{R^2}, 0 \right)$$

for the magnetic field due to a straight wire, lying along the $z-$axis and carrying current I (Here $R^2 = x^2 + y^2$).