

Numerical Solution of Differential Equations II. QS 1 (HT 2020)

1. By (artificially) employing analytic solution methods for initial value problems (but not for the BVP) solve

$$u'' - 2u' + 2u = 2, \quad u(0) = 1, \quad u\left(\frac{\pi}{2}\right) = 2$$

by the shooting method for linear problems. (You may find that selection of initial slopes $u'_0(0) = 0$ and $u'_1(0) = 1$ where u_0 solves the inhomogeneous IVP and u_1 solves the homogeneous IVP makes the calculation a little easier, although any values should work!)

2. As Qn 1 above; why would there be a difficulty if the right hand boundary condition was set to $u(\pi) = 2$? What solution(s) exist in this case?

3. Show that solving

$$u'' = 2u' - 2u + 2, \quad u(0) = 1, \quad u\left(\frac{\pi}{2}\right) = 2$$

by the general method for nonlinear equations gives the exact initial slope which gives the solution of the BVP at the first Newton iteration when $u'(0) = s = 0$ is chosen. Is this true for any other values of s ?

4. Use the shooting method to solve the BVP

$$u'' = 100u, \quad u(0) = 1, \quad u(3) = \epsilon + \cosh 30.$$

If the linearly independent solutions

$$u_0(x) = \cosh 10x, \quad u_1(x) = \frac{1}{10} \sinh 10x$$

are employed, show that

$$\mu = 10\epsilon / \sinh 30.$$

Explain the difficulty in carrying out the numerical solution for small $|\epsilon|$.

5. For the problem

$$(pu')' + qu = f, \quad u(a) = \alpha, \quad u(b) = \beta,$$

approximated by

$$\frac{p(x_{j+\frac{1}{2}})u_{j+1} - \left[p(x_{j+\frac{1}{2}}) + p(x_{j-\frac{1}{2}}) \right] u_j + p(x_{j-\frac{1}{2}})u_{j-1}}{h^2} + q(x_j)u_j = f(x_j),$$

$$j = 1, \dots, n, \quad u_0 = \alpha, \quad u_{n+1} = \beta,$$

show that the local truncation error is $\mathcal{O}(h^2)$ provided $u(x)$ and $p(x)$ have four derivatives.

[Note: $p(x_{j\pm\frac{1}{2}}) = p(x_j) \pm \frac{h}{2}p'(x_j) + \frac{1}{8}h^2p''(x_j) \pm \frac{h^3}{48}p'''(x_j) + \mathcal{O}(h^4)$.]