Numerical Solution of Differential Equations II. QS 2 *(HT 2020)*

1. If A is SCDD ($|a_{jj}| > \sum_{\substack{i=1\\ i\neq j}}^{n} |a_{ij}|$ each j) and after the first step of Gaussian Elimination it is reduced to

$$\begin{bmatrix} a_{11} & a_{12}\dots a_{1n} \\ 0 & B \end{bmatrix},$$

prove that B is SCDD. Deduce that for SCDD A, all multipliers used in Gaussian Elimination have absolute value less than 1.

2. The Poisson problem $\nabla^2 u = f$ is to be solved on a spherically symmetric region $a \le r \le b$ leading to the problem of finding $u(r)$ which satisfies:

$$\frac{1}{r^2}\frac{d}{dr}(r^2\frac{du}{dr}) = f(r), \quad a < r < b, \quad u(a) = \alpha, \quad u(b) = \beta.$$

Write down the finite difference approximation as in Qu. 1 above (i.e. $(pu')' = r^2 f$ with $r^2 = p$) on a regular grid of size h : check that the resulting matrix is symmetric tridiagonal.

3. In Qu. 2 above, if the PDE is now $\nabla^2 u - u = f$, prove that the matrix resulting is SRDD for any finite h (and hence symmetric $\Longrightarrow$ SCDD).

4. Show that approximation of

$$(pu')' = pu'' + p'u' = f, \quad u(a) = \alpha, \quad u(b) = \beta,$$

by the finite difference approximation

$$p(x_j)\frac{U_{j+1} - 2U_j + U_{j-1}}{h^2} + p'(x_j)\frac{U_{j+1} - U_{j-1}}{2h} = f(x_j), \quad U_0 = \alpha, \quad U_{n+1} = \beta$$

yields a tridiagonal but <u>not</u> symmetric matrix.

5. The equation

$$u'' - au' - bu = g, \quad 0 \le x \le 1,$$

(where a, $b > 0$, g are suitably differential functions), is discretised using central differences on a uniform mesh of size h.

Show that the coefficient matrix for the discrete problem is diagonally dominant provided

$$h < \frac{2}{\max|a|}.$$