Numerical Solution of Differential Equations II. QS 4 *(HT 2020)*

1. Suppose Ω is the right-angled isosceles triangle

$$\{(x,y)\ :\ 0 \le x \le 1,\ \ 0 \le y \le 1-x\}$$

and $\delta\Omega$ its boundary. Use a regular grid on this domain with $x_0 = 0$, $x_j = jh$, $(n{+}1)h = 1$, $y_0 = 0$, $y_k = kh$ and apply the 5-point finite difference formula to approximate the solution of $-\nabla^2 u = f$ on Ω with $u = 0$ on $\delta\Omega$. How many rows of the coefficient matrix have only 2 non-zero entries? How many have 3, 4 non-zero entries? What is the structure of the coefficient matrix with Lexicographic ordering? (This is not so easy as for a square! Do it for $n = 5$ if you prefer.) Is this matrix Irreducibly Diagonal Dominant?

2. Verify that the matrix obtained for central difference approximation of

$$u'' + qu = f \quad \text{on} \quad [a,b]\,, \quad u(a) = \alpha, \quad u'(b) = \beta,$$

where $q \in \mathbb{R}$ is a constant, is Irreducibly Diagonally Dominant for every (small) mesh size h if and only if $q \le 0$.

3. If

$$L_h U_{j,k} = \frac{1}{h^2}\left(4U_{j,k} - U_{j+1,k} - U_{j-1,k} - U_{j,k+1} - U_{j,k-1}\right),$$

show that $L_h \Psi_{j,k} = -4$, where $\Psi_{j,k} = (x_j - \frac{1}{2})^2 + (y_k - \frac{1}{2})^2$. Hence, by using the minimum principle on $\phi_{j,k} = e_{j,k} - \frac{1}{4}\tau\Psi_{j,k}$, prove that $-\frac{\tau}{8} \le e_{j,k}$, $\ j,k = 1,\ldots,n$.

4. For the rotated 5-point approximation of $-\nabla^2 u = f$ with Dirichlet boundary conditions on the unit square and with lexicographic ordering, what is the form of the coefficient matrix? Is it irreducibly diagonally dominant?