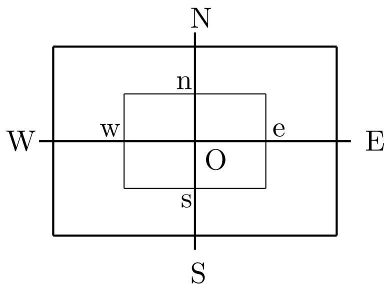
Numerical Solution of Differential Equations II. QS 5 *(HT 2020)*

1. Suppose the rectangular region $\Omega = [0, H] \times [0, L] \in \mathbb{R}^2$ is covered with a mesh of n points in each of the x and y coordinate directions where $h = \frac{H}{n+1}$ is the mesh size in the x direction and $l = \frac{L}{n+1}$ is the mesh size in the y direction.

If the problem $\frac{\partial}{\partial x}\left(p\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(p\frac{\partial u}{\partial y}\right) = f$ on Ω with u given on $\partial\Omega$ is to be solved with p a continuous function which does not change sign in $\partial\Omega$, derive the formula

$$\frac{h}{l}p(s)U_S+\frac{l}{h}p(e)U_E+\frac{h}{l}p(n)U_N+\frac{l}{h}p(w)U_W-\left(\frac{h}{l}p(s)+\frac{l}{h}p(e)+\frac{h}{l}p(n)+\frac{l}{h}p(w)\right)U_0 = hlf(0)$$

from applying the Divergence Theorem to the indicated rectangular area and using simple approximation for the line integrals which result.



Verify that in the special case $p = -1$ and $h = l$ this is the usual 5-point formula. For the non-special case, is the matrix symmetric?

2. Consider the Neumann Problem for the Laplace Equation

$$-\nabla^2 u = 0 \text{ in } (0,1)\times(0,1) = \Omega, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega.$$

(a) If the boundary condition is approximated by central differences, eg. $\frac{U_{-1,k}-U_{1,k}}{2h} = 0$ on the left side and correspondingly on the other 3 sides, and the fictitious values (e.g. $U_{-1,k}$) are eliminated from the standard 5-point finite difference stencil which is used in the interior and up to the boundary, write down the structure of the resulting $(n+2)^2\times(n+2)^2$ matrix, A, which results under the usual lexicographic ordering.

(b) Is A irreducibly diagonally dominant?

(c) Is A non-singular?

(d) Show that the vector v^{rs}, $r, s = 0, \ldots, n+1$ with entries

$$v^{rs}_{jk} = \cos\frac{jr\pi}{n+1}\cos\frac{ks\pi}{n+1}$$

is and eigenvector of A : find the corresponding eigenvalue.

(e) Comment on v^{00}.

(f) Show that fixing the value of any particular U_{jk} on $\partial\Omega$ (which corresponds to applying a Dirichlet boundary condition at one point) will make the finite difference solution unique.

3. Calculate the local truncation error for the Lax-Wendroff finite difference scheme for the first order wave equation. What is the CFL stability criteria for this scheme.

4. By writing the scheme in the form

$$U_j^{n+1} = \mathcal{H}_k U^n,$$

prove that the first order upwind scheme is stable for the first order wave equation provided

$$\left|\frac{ak}{h}\right| \leq 1.$$

You should use the discrete norm defined by

$$\|U^n\| = h\sum_j |U_j^n|.$$