Question Sheet 4

1. Briefly derive the *shallow water equations*

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0, \qquad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0,$$

outlining the assumptions under which they hold.

Deduce from these equations that $u \pm 2c$ is constant along curves satisfying $\mathrm{d}x/\mathrm{d}t = u \pm c$, where $c = \sqrt{gh}$.

Shallow water of depth h_0 is held at rest in $x > 0$ by a dam at $x = 0$. At $t = 0$ the dam starts to leak so that water flows through with a volume flux proportional to the net force on the dam.

Deduce that $u = -kc^2$ at $x = 0$, for some positive constant k. Hence show that, provided $c_0 k < 3/2$, where $c_0 = \sqrt{gh_0}$, the depth is given by

$$\sqrt{gh} = c = \begin{cases} \dfrac{-1+\sqrt{1+2kc_0}}{k} & 0 < \dfrac{x}{t} < c_0 - \dfrac{3}{k}\left(1 + c_0 k - \sqrt{1+2c_0 k}\right), \\ \dfrac{1}{3}\left(2c_0 + \dfrac{x}{t}\right) & c_0 - \dfrac{3}{k}\left(1 + c_0 k - \sqrt{1+2c_0 k}\right) < \dfrac{x}{t} < c_0, \\ c_0 & \dfrac{x}{t} > c_0. \end{cases}$$

2. (a) The Rankine–Hugoniot conditions for a stationary planar shock are

$$\left[\rho u\right]_-^+ = \left[p + \rho u^2\right]_-^+ = \left[\frac{u^2}{2} + \frac{\gamma p}{(\gamma - 1)\rho}\right]_-^+ = 0.$$

Find a relationship between the upstream and downstream Mach numbers, M_- and M_+, and show that the density and pressure ratios across the shock are given by

$$\frac{\rho_+}{\rho_-} = \frac{(\gamma+1)M_-^2}{2 + (\gamma - 1)M_-^2}, \qquad \frac{p_+}{p_-} = 1 + \frac{2\gamma}{\gamma+1}\left(M_-^2 - 1\right),$$

where subscripts $-$ and $+$ refer to upstream and downstream values respectively.

Deduce that $p_+/\rho_+^\gamma > p_-/\rho_-^\gamma$ if and only if $M_- > 1$.

Show also that the temperature of the gas increases on passing through the shock.

(b) A gas with speed of sound c_0 flows steadily at speed U in a long, straight, uniform duct. The end of the duct is suddenly closed, so that the gas is brought to rest over a continuously increasing length of the duct. Show that the shock wave so caused propagates upstream with a speed V given by

$$V = \frac{1}{4}\left\{\sqrt{(1+\gamma)^2 U^2 + 16c_0^2} - (3-\gamma)U\right\}.$$

3. (a) Using the principles of conservation of mass and momentum, derive the Rankine–Hugoniot conditions

$$\left[hu\right]_-^+ = \left[hu^2 + \frac{gh^2}{2}\right]_-^+ = 0 \qquad (\dagger)$$

satisfied across a stationary hydraulic jump.

(b) Show that for a given mass flux $hu = q$, and depth h_- on one side of the jump, the depth on the other side h_+ is uniquely defined. Show further that one of h_- and h_+ must be larger than $q^{2/3}/g^{1/3}$, while the other is smaller.

(c) Show that energy is produced by the jump at a rate

$$Q = \left[\rho hu\left(\frac{u^2}{2} + gh\right)\right]_-^+ = \frac{\rho g u_-(h_- - h_+)^3}{4h_+}.$$

Deduce that the depth must increase as the fluid passes through the jump.

(d) Explain how (†) must be modified to describe a bore moving at speed V.

(e) A bore separates water of depth h_- in $x < Vt$ from stationary water of depth $h_+ < h_-$ in $x > Vt$. Show that the bore speed is given by

$$V = \pm\sqrt{\frac{gh_-(h_+ + h_-)}{2h_+}}.$$

How should the sign of V be chosen?

4. A stationary shock in a two-dimensional ideal gas flow is at $x = 0$. In the region $x < 0$ the gas is uniform with pressure p_-, density ρ_- and velocity (u_-, v_-), where $u_- > 0$ and $v_- > 0$. In the region $x > 0$ the corresponding quantities are p_+, ρ_+, u_+ and v_+.

Using conservation of mass, momentum and energy, show that $v_+ = v_-$ and derive the Rankine–Hugoniot conditions in the form

$$\left[\rho u\right]_-^+ = \left[p + \rho u^2\right]_-^+ = \left[\frac{u^2}{2} + \frac{\gamma p}{(\gamma - 1)\rho}\right]_-^+ = 0.$$

What thermodynamic principle may be used to show that $\rho_+ > \rho_-$?

Sketch a streamline passing through the shock, indicating the direction of deflection. Show that the angle of deflection δ is given by

$$\tan\delta = \frac{u_- - u_+}{v_- + u_- u_+/v_-}$$

and, by maximising the right-hand side as a function of v_-, deduce that

$$\tan\delta \leqslant \frac{1}{2}\left(\sqrt{\frac{u_-}{u_+}} - \sqrt{\frac{u_+}{u_-}}\right).$$

Using the results from question 2 above, show that

$$\frac{\gamma - 1}{\gamma + 1} < \frac{u_+}{u_-} < 1\,.$$

Hence deduce that

$$\tan\delta < \frac{1}{\sqrt{\gamma^2 - 1}}.$$