B5.4 Waves & Compressible Flow

Question Sheet 3 Solutions

(i) (a) Define

$$I = \int_{-R}^{R} \mathrm{e}^{\pm \mathrm{i} s^2}\, \mathrm{d}s, \tag{1}$$

and deform the contour into three components (see the figure):

$$\gamma_1 : s = R\mathrm{e}^{i(\theta+\pi)}, \quad \gamma_2 : s = \frac{(1 \pm i)\, t}{\sqrt{2}}, \quad \gamma_3 : s = R\mathrm{e}^{i\theta}. \tag{2}$$

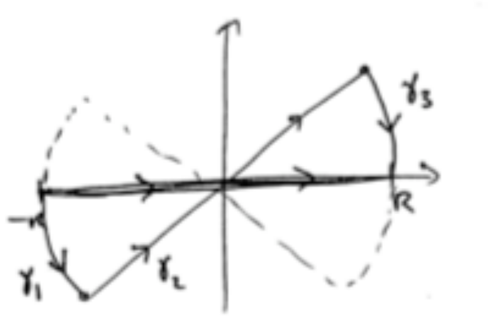

Figure 1: Contours for question (i)

Then,

$$\begin{aligned} I =& \int_0^{\pm\pi/4} \mathrm{e}^{iR^2(\cos 2\theta + i \sin 2\theta)} 2iR\mathrm{e}^{i(\theta+\pi)}\, \mathrm{d}\theta + \int_{-R}^{R} \mathrm{e}^{(\pm i)^2 t^2} \frac{(1 \pm i)}{\sqrt{2}}\, \mathrm{d}t \\ &+ \int_0^{\pm\pi/4} \mathrm{e}^{\pm iR^2(\cos 2\theta + i\sin 2\theta)} 2iR\mathrm{e}^{i\theta}\, \mathrm{d}\theta, \\ =& I_1 + I_2 + I_3. \end{aligned} \tag{3}$$

Now,

$$I_1 = \int_{-\infty}^{\infty} \mathrm{e}^{-t^2}\, \mathrm{d}t \frac{(1 \pm i)}{\sqrt{2}} = \sqrt{\pi}\frac{(1 \pm i)}{\sqrt{2}} = (1 \pm i)\sqrt{\frac{\pi}{2}}. \tag{4}$$

Also,

$$I_2 = -2\int_R^{\infty} \mathrm{e}^{-t^2}\, \mathrm{d}r \frac{(1 \pm i)}{\sqrt{2}}, \tag{5}$$

so that (substituting $t = R + s$),

$$|I_2| \le 2\mathrm{e}^{-R^2} \int_0^{\infty} \mathrm{e}^{-Rs}\mathrm{e}^{-s^2}\, \mathrm{d}s \le 2\mathrm{e}^{-R^2}\frac{\sqrt{\pi}}{2} = \mathcal{O}\left(\mathrm{e}^{-R^2}\right). \tag{6}$$

Finally,

$$I_3 = 4iR\int_0^{\pm\pi/4} \mathrm{e}^{\pm iR^2(\cos\theta + i\sin 2\theta)}\mathrm{e}^{i\theta}\, \mathrm{d}\theta, \tag{7}$$

so that (using $\sin 2\theta \geq 2\theta/\pi$),

$$
\begin{aligned}
|I_3| &\leq 4R \int_0^{\pm\pi/4} \left| \mathrm{e}^{\pm R^2 \sin 2\theta} \right| \mathrm{d}\theta \\
&\leq 4R \int_0^{\pi/4} \mathrm{e}^{-R^2 \cdot 2\theta/\pi} \, \mathrm{d}\theta, \\
&\leq 4R \frac{\pi}{2R^2} \left[\mathrm{e}^{-2R^2\theta/\pi} \right]_0^{\pi/4}, \\
&= \mathcal{O}\left(1/R\right). \qquad (8)
\end{aligned}
$$

Hence,

$$I = (1 \pm i) \sqrt{\frac{\pi}{2}} + \mathcal{O}\left(1/R\right). \qquad (9)$$

(b) Define

$$I(t) = \int_a^b f(k) \mathrm{e}^{\mathrm{i}kt} \, \mathrm{d}k. \qquad (10)$$

We integrate by parts by setting $u = \mathrm{e}^{ikt}/it$ and $v = f\left(k\right)$ to give

$$I\left(t\right) = \left[\frac{1}{it} \mathrm{e}^{ikt} f\left(k\right) \right]_a^b - \int_a^b \frac{1}{it} \mathrm{e}^{ikt} f'\left(k\right) \, \mathrm{d}k. \qquad (11)$$

Therefore

$$\left|I\left(t\right)\right| \leq \frac{1}{t} \left(\left|f\left(a\right)\right| + \left|f\left(b\right)\right| \right) + \frac{1}{t} \left(b-a\right) \sup_k \left\{ \left|f'\left(k\right)\right| \right\}, \qquad (12)$$

and providing f and f' are bounded,

$$\left|I\left(t\right)\right| \leq \frac{M}{t} \quad \text{for some } M. \qquad (13)$$

Therefore $I\left(t\right) = \mathcal{O}\left(1/t\right)$ as $t \to \infty$.

(c) See lecture notes.

(ii) The Fourier transform of $\epsilon/(x^2 + \epsilon^2)$ is given by

$$\hat{f}\left(k\right) = \int_{-\infty}^{\infty} \frac{\epsilon}{x^2 + \epsilon^2} \mathrm{e}^{-ikx} \, \mathrm{d}x. \qquad (1)$$

The integrand has poles at $x = \pm i\epsilon$, and can be written in the form

$$\frac{\epsilon \mathrm{e}^{-ikx}}{\left(x + i\epsilon\right)\left(x - i\epsilon\right)}. \qquad (2)$$

Therefore, the residues are

$$\frac{\epsilon \mathrm{e}^{\pm k\epsilon}}{\pm 2i\epsilon} = \mp \frac{1}{2} i \mathrm{e}^{\pm k\epsilon}. \qquad (3)$$

For $k > 0$ we close the contour in the lower half plane, and for $k < 0$ we close the contour in the upper half plane (see figure 2). Jordan's Lemma tells us that the semi-circular contour doesn't contribute as $R \to \infty$, so we just pick up the residue at the pole. Therefore,

$$\hat{f}\left(k\right) = \left\{ \begin{array}{ll} \left(-2\pi i\right) \left(\frac{i}{2} \mathrm{e}^{-k\epsilon}\right) & \text{for } k > 0, \\ \left(2\pi i\right) \left(-\frac{i}{2} \mathrm{e}^{k\epsilon}\right) & \text{for } k < 0, \end{array} \right. , \qquad (4)$$

so that

$$\hat{f}(k) = \pi \mathrm{e}^{-|k|\epsilon}. \tag{5}$$

Also note that the case $k = 0$ follows by continuity, or by direct integration:

$$\hat{f}(0) = \int_{-\infty}^{\infty} \frac{\epsilon}{x^2 + \epsilon^2}\, \mathrm{d}x = \left[\arctan\left(\frac{x}{\epsilon}\right)\right]_{-\infty}^{\infty} = \pi. \tag{6}$$

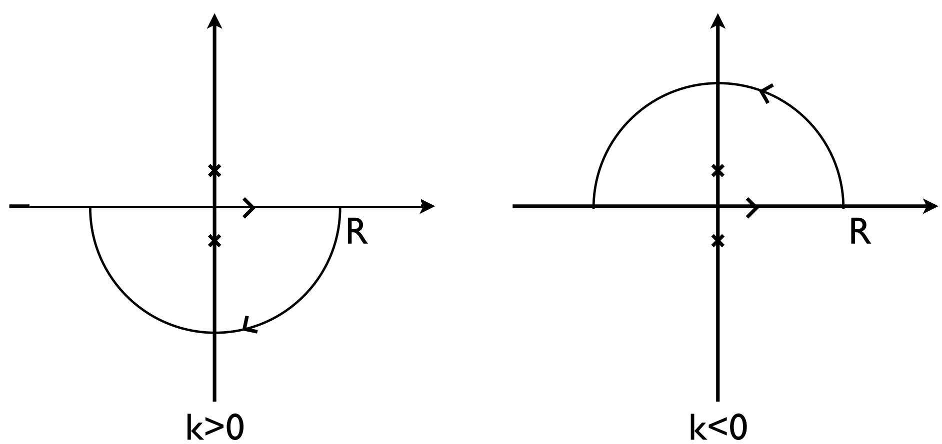

Figure 2: The contours for Question (ii).

Now we assume that fluid occupying the half-space $z < 0$ starts from rest with the initial free surface profile $\eta_0(x) = -a\epsilon/\pi(x^2 + \epsilon^2)$. The solution in the lecture notes gives

$$\hat{\eta}(k,t) = \hat{\eta}_0(k) \cos(w(k)\, t), \tag{7}$$

with $w(k) = \sqrt{g|k|}$. Using the result above we have that $\hat{\eta}_0(k) = -a\mathrm{e}^{-\epsilon|k|}$. We invert the transform to get

$$\begin{aligned} \eta(x,t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\eta}(k,t)\, \mathrm{e}^{ikx}\, \mathrm{d}k, \\ &= -\frac{a}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\epsilon|k|} \cos\left(t\sqrt{g|k|}\right) \mathrm{e}^{ikx}\, \mathrm{d}k, \\ &= I_+ + I_-, \end{aligned} \tag{8}$$

where

$$I_\pm = -\frac{a}{4\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\epsilon|k|} \mathrm{e}^{i\left(kx/t \pm \sqrt{g|k|}\right)}\, \mathrm{d}k. \tag{9}$$

The method of stationary phase tells us that the main contribution to the integral comes from $\psi'(k) = 0$ where

$$\psi(k) = \frac{kx}{t} \pm \sqrt{g|k|}, \tag{10}$$

i.e. where

$$\frac{x}{t} \pm \frac{\sqrt{g|k^\star|}}{2k^\star} = 0 \quad \Rightarrow \quad k^\star = \mp \frac{gt^2}{4x^2}. \tag{11}$$

Further,

$$\psi''(k^\star) = \mp \frac{\sqrt{g|k^\star|}}{4k^{\star 2}} = \mp \frac{\sqrt{g}}{4} \frac{8x^3}{g^{3/2}t^3} = \mp \frac{2x^3}{gt^3}. \tag{12}$$

Therefore the method of stationary phase gives

$$I_\pm \sim -\frac{a}{4\pi} \mathrm{e}^{-\epsilon|k^\star|} \mathrm{e}^{i\left(\psi(k^\star)t + \frac{\pi}{4}\mathrm{sgn}(\psi''(k^\star))\right)} \sqrt{\frac{2\pi}{|\psi''(k^\star)|\,t}}. \tag{13}$$

Now, for ϵ sufficiently small, $\mathrm{e}^{-\epsilon|k^\star|} \sim 1$, so

$$\begin{aligned} I_\pm &\sim -\frac{a}{4\pi} \mathrm{e}^{i\left[\mp\frac{gt^2}{4x} \pm \frac{gt^2}{2x} \mp \frac{\pi}{4}\right]} \sqrt{\frac{2\pi g t^3}{2x^3 t}} \\ &\sim -\frac{at}{4} \mathrm{e}^{\pm i\left[\frac{gt^2}{4x} - \frac{\pi}{4}\right]} \sqrt{\frac{g}{\pi x^3}}. \end{aligned} \tag{14}$$

Therefore,

$$\eta(x,t) = I_+ + I_- \sim -\frac{at}{2} \sqrt{\frac{g}{\pi x^3}} \cos\left(\frac{gt^2}{4x} - \frac{\pi}{4}\right). \tag{15}$$